Part I. Compressible flow (cont'd)

4. Nonlinear systems of conservation laws

If the system of equations under consideration is nonlinear, our task is much more difficult, but if Allah is with us, not intractable. We would like to solve an initial-value problem for the vector $U(x,t) \in \Re^N$ with $F(U) \in \Re^N$ such that:

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0 \tag{1}$$

with $U(x,0) = U_o$ given and F is now a nonlinear function of U. We can still talk about hyperbolicity, although we will require some special restrictions. As a first attempt, we will try to do the same thing as we did with linear systems, i.e., diagonalize the system and decouple the equations.

If F is differentiable, we can still say that:

$$A(U_o) = \nabla_U F = \left.\frac{\partial F_i}{\partial U_j}\right|_{U = U_o} \tag{2}$$

and put the PDE in quasilinear form

$$\frac{\partial U}{\partial t} + A(U)\frac{\partial U}{\partial x} = 0 \tag{3}$$

We require that A has n linearly independent right eigenvectors r_k:

$$Ar_k = \lambda_k r_k \tag{4}$$

where λ_k are real eigenvalues. We further require strictly ascending ordering on eigenvalues:

$$\lambda_1 < \ldots < \lambda_k < \ldots < \lambda_n \tag{5}$$

for the right eigenvectors $r_1 \ldots r_k \ldots r_n$.

We can still define R to be a rectangular matrix made up of columns of n right eigenvectors:

$$R = (r_1, \ldots, r_k, \ldots, r_n) \tag{6}$$

and a left-eigenvector matrix, L, analogous to R, which is made up of n rows of left eigenvectors which satisfy:

$$l_k A = \lambda_k l_k \tag{7}$$

so that L looks like:

$$L = R^{-1} = \begin{bmatrix} l_1 \\ \dots \\ l_n \end{bmatrix} \tag{8}$$

Even though the structure of the the system PDE's and associated eigenvalues and eigenvectors begins to look familiar, we must handle them in a fundamentally different way from the linear system, since everything is now nonlinear, i.e., $A = A(U)$ which implies that $\lambda_k = \lambda_k(U)$; $r_k = r_k(U)$; and $l_k = l_k(U)$. We begin with a negative statement: we can not diagonalize the system by:

$$U(x, t) = \sum \alpha_k (x, t)\, r_k (x, t) \tag{9}$$

where

$$\alpha_k (x, t) = l_k U (x, t) \tag{10}$$

To show this, substitute the expansion for U above in equation (-2) into the quasilinearized PDE of equation (-8):

$$\frac{\partial}{\partial t} \sum \alpha_k r_k + A(U) \frac{\partial}{\partial x} \sum \alpha_k r_k = 0$$

$$\sum_k \left(\frac{\partial \alpha_k}{\partial t} r_k + \frac{\partial r_k}{\partial t} \alpha_k \right) + A(U) \sum_k \left(\frac{\partial \alpha_k}{\partial x} r_k + \frac{\partial r_k}{\partial x} \alpha_k \right) = 0$$

Premultiply by l_k and use equation (-7) to substitute for $l_k A$:

$$\sum_k \left(l_k \cdot \left(r_k \frac{\partial \alpha_k}{\partial t} + \lambda_k r_k \frac{\partial \alpha_k}{\partial x} \right) \right) + \sum_k \alpha_k l_k \left(\frac{\partial r_k}{\partial t} + A \frac{\partial r_k}{\partial x} \right) = 0 \tag{11}$$

(Double-check this equation)

The second term on the right presents trouble since it has a derivative of the eigenvector with respect to U in it:

$$\frac{\partial r_k}{\partial t} = \nabla_U r_k \frac{\partial U}{\partial t} \tag{12}$$

For special systems, one may be able to find a nonlinear change of variables which will do what we want. For a 2x2 system, these variables are the Riemann invariants. The general statement for nonlinear systems is that signals can be interpreted as increments or perturbations of some background solution. Also, these increments propagate along characteristics. We can then look for weak solutions which satisfy the conditions associated with each individual increment. So, we will study single-wave solutions separated by constant states, i.e., the analogs of isolated shocks. This means that $U(x,t)$ is a function only of x/t. Then we will piece together N individual waves to solve the Riemann problem, as shown in figure +1. This is certainly the simplest way of approaching this problem, although it is not the only one. However, if this technique finds an answer, we can stop searching for alternate solution methods since there is a unique solution.

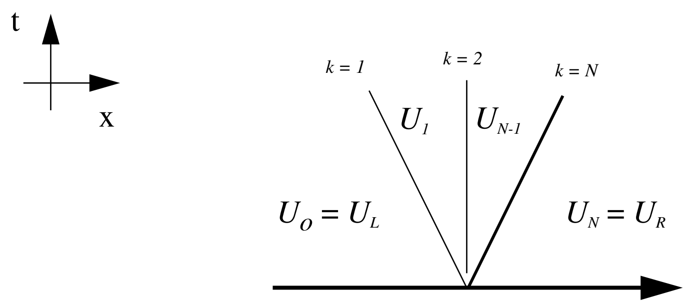


FIGURE 1. Graphical representation of approach to a nonlinear system as a sequence of constant states separated by discontinuities.

4.1 The Riemann problem

The constant states U_o, U_1, ... U_N are each separated by small jumps. The weak-solution criterion can be used to tell us what the jump must satisfy. The Rankine-Hugoniot relation for the *k*th discontinuity is:

$$F(U_{k-1}) - F(U_k) = s(U_{k-1} - U_k) \tag{13}$$

These things look like signals propagating. To solve this problem, we need to find (U_k - U_{k-1}) and the speed at which the signals travel. We assume that nonlinear waves are weakly nonlinear, so that in the limit of small U_k - U_{k-1}:

$$U_k - U_{k-1} = \alpha_k r_k \tag{14}$$

The quantity $\alpha_k r_k$ can be interpreted as the *increments* of change in U. The information which propagates along characteristics are the amplitude of the jumps α_k; the direction in which it propagates is given by r_k.

4.2 The entropy condition

We are still missing a statement of the entropy condition for the case of nonlinear systems. We hope that this final piece of the puzzle, along with the weak-solution criterion, will guarantee a unique solution to the problem. So, to formulate an analog of the Lax entropy condition, we need an analogy with convex functions. First, we require that admissible solutions will still be the limit of viscous solutions. As for the nonlinear scalar case, if the viscous solution U^{ε} is defined by:

$$\frac{\partial U^{\varepsilon}}{\partial t} + \frac{\partial F(U^{\varepsilon})}{\partial x} = \varepsilon \frac{\partial^2 U^{\varepsilon}}{\partial x^2} \tag{15}$$

then we want U to satisfy:

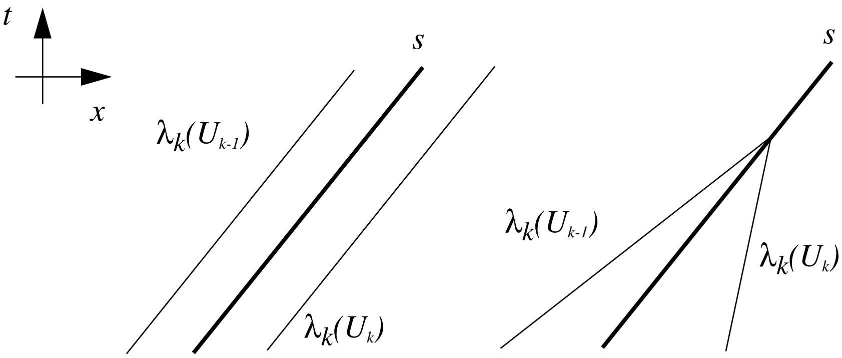
$$U = \lim_{\varepsilon \to 0} U^{\varepsilon} \tag{16}$$

For the nonlinear scalar case, $a = a(u)$. If a increased with increasing u, $\partial a / \partial u > 0$ and $f'' > 0$. This meant that as the strength of the wave increased, the jump in a also increased. The nonlinear system analog to a will be the speed of the disturbance propagation, λ_k. The strength of the wave is measured by the amplitude α_k. So, for nonlinear systems, the quantity we want to examine in order to satisfy the entropy conditions is $\partial\lambda_k / \partial u$. Specifically, we want $\partial\lambda_k / \partial u > 0$. Given $\lambda_k(U)$ and $r_k \cdot \nabla_U \lambda_k$, we focus on on the kth discontinuity.

Two cases are possible.

(1) **Linearly degenerate:** For this case, $r_k \cdot \nabla_U \lambda_k \equiv 0$ for all U. The entire kth wave family is said to be linearly degenerate. Note that by definition, r_k must be perpendicular to $\nabla_U \lambda_k$, which looks like the linear scalar case. This is depicted graphically in figure +1(a).

(2) **Genuinely nonlinear:** For this case, $r_k \cdot \nabla_U \lambda_k \neq 0$ for all U. Then the kth wave family is said to be genuinely nonlinear. This means that r_k must never be perpendicular to $\nabla_U \lambda_k$, as shown in figure +1(b).



(a) Linearly degenerate (*LD*) **(b) Genuinely nonlinear (*GN*)**

FIGURE 2. Linearly degenerate and genuinely nonlinear wave families.

We note that

$$U_k - U_{k-1} = \alpha_k r_k$$

so that

$$\lambda_k(U_k) - \lambda_k(U_{k-1}) = \lambda_k(U_{k-1} + \alpha_k r_k) - \lambda_k(U_{k-1})$$

Using a Taylor expansion

$$\lambda_k(U_k) - \lambda_k(U_{k-1}) = \alpha_k (r_k \cdot \nabla_U \lambda_k) \tag{17}$$

For the genuinely nonlinear case, we know that the r.h.s. of equation (-1) is not zero. This would indicate that as something crosses the discontinuity, its speed changes.

We will assume that waves are either linearly degenerate (*LD*) or genuinely nonlinear (*GN*). Whether or not a wave is *GN* or *LD* depends on the wave family k. In the total solution to the Riemann problem, some of the waves may be *GN* while others may be *LD*. However, for each k, the kth family is either one or the other. We know that the entropy condition corresponding to the Lax entropy condition is always satisfied for *LD* waves. For *GN* waves, we will add the restriction that

$$\lambda_k(U_{k-1}) > s > \lambda_k(U_k) \tag{18}$$

This means that characteristics must converge to the shock, i.e., no entropy-violating rarefaction shocks are allowed.

Secondly, we will also assume that there are no coinciding values of λ_k. This is necessary because ...

4.3 Solution procedure for the approximate Riemann problem

We would like to solve the Riemann problem for a nonlinear system of equations. We are given initial data on the left and right states:

$$U(x, 0) = \begin{cases} U_L = U_{k-1} \ldots if \ldots (x < 0) \\ U_R = U_k \ldots if \ldots (x > 0) \end{cases} \tag{19}$$

as shown in figure -1 previously. We make the following assumptions:

- *U(x, t)* is a function of *x/t* only;
- *U(x, t)* is the limit of the viscous solution, thus satisfying the entropy conditions;
- For any *k*th wave, the wave families are either linearly degenerate *(LD)* or genuinely nonlinear *(GN)*.

The solution procedure is as follows:

4.3.1 Phase space

The first step in the solution procedure is to find the phase space solution U_k.

To get the ball rolling, we need to compute the left and right eigenvector matrices, *L* and *R* at a point. As a logical starting place, calculate them as functions of $(U_L + U_R)/2$. There will be a number of intermediate constant states in between these left and right states with incremental discontinuities between them. Graphically:

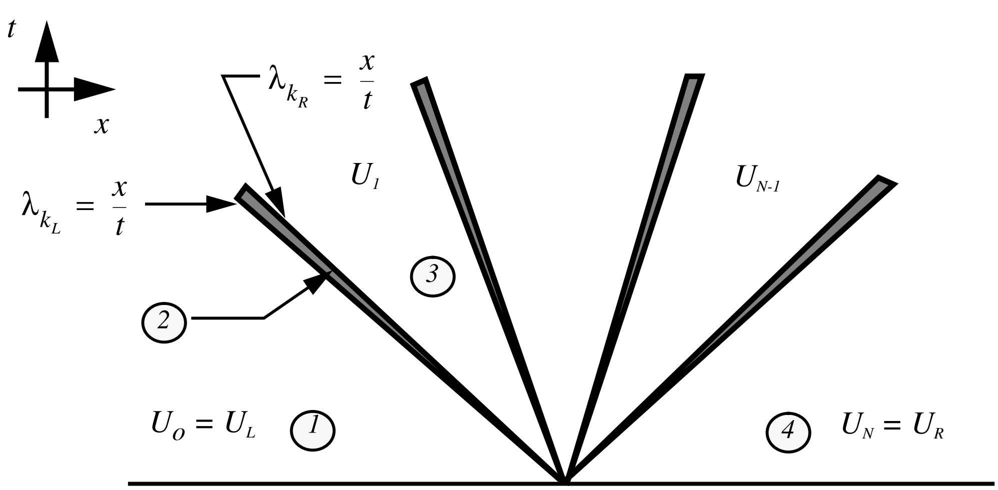


FIGURE 3. Separation of the problem into a series of constant states separated by incremental discontinuities.

For the *k*th wave:

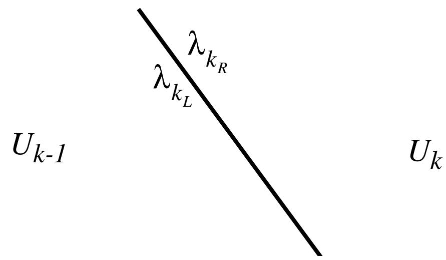


FIGURE 4. Left and right states and wavespeeds for the kth wave

We define the wave amplitudes by:

$$\alpha_k = l_k \cdot (U_R - U_L) \tag{20}$$

or, equivalently:

$$U_R - U_L = \sum \alpha_k r_k \tag{21}$$

We have evaluated the eigenvectors and amplitudes at the average of the left and right states, $(U_L + U_R)/2$. If the *k*th wave is *LD* and $\lambda_k(U_{k-1}) < \lambda_k(U_k)$, the eigenvalues are defined by:

$$\lambda_{k_L} = \lambda_k(U_{k-1}) \tag{22}$$

$$\lambda_{k_R} = \lambda_k(U_k) \tag{23}$$

Otherwise:

$$\lambda_{k_L} = \lambda_{k_R} = \frac{1}{2}\lambda_k(U_{k-1} + U_k) \tag{24}$$

4.3.2 Physical space

Given the value of *x/t*, we can now find *U(x, t)*. Recall that U_k can be found by:

$$U_k = U_L + \sum_{k' \le k} \alpha_{k'} r_{k'} \tag{25}$$

There is a jump carried with each wave. Consider the *k*th wave. If it is linearly degenerate (*LD)*, treat it as a discontinuity traveling at the speed $s = [\lambda_k(U_{k-1}) + \lambda_k(U_k)]/2$. If it is genuinely nonlinear *(GN)*, treat it as a discontinuity which satisfies the analog of the Lax entropy condition. Four cases are possible, as shown in figure -2 above:

(1) $x/t < \lambda_{k_L}$: In this case, $U(x, t) = U_L$, since we are downstream of the wave.

(2) $\lambda_{k_L} \le x/t < \lambda_{k_R}$: This corresponds to a rarefaction fan. Linearly interpolate to get inside the rarefaction wave.

$$U(x,t) = U_{k-1} + \frac{x/t - \lambda_{k_L}}{\lambda_{k_R} - \lambda_{k_L}} [U_k - U_{k-1}] \ldots k = 1, \ldots, N$$

(3) $\lambda_{k-1_R} \le x/t < \lambda_{k_L}$: For this case, $U(x,t) = U_{k-1} \ldots k = 1, \ldots, N$.

(4) $x/t > \lambda_{k_R}$: Simply, $U(x,t) = U_R$.

4.3.3 Miscellaneous tricks

Although this is a nonlinear problem, it looks very similar to the linear case. The key is being sure that the entropy condition is not violated, so we must check carefully for *GN* waves and rarefaction shocks. To be smart about this, reach into our bag of tricks and pull out the following assorted procedures:

(1) **Choose the right nonlinear change of variables.** Work in terms of the variables *V = V(U),* e.g., the "primitive" variables for gasdynamics

$$V(U) = \begin{bmatrix} \rho \\ u \\ p \end{bmatrix} \tag{26}$$

The goal is to find a change of variables which makes it easier to compute the eigenvectors, especially the right-eigenvector matrix, *R*. It should also be simpler to verify the ordering of the wavespeeds λ_k. For a more complete discussion, see section xxx.xx.x.

(2) **Augment the dependent variables.** For the case of gasdynamics, the nonlinear change of variables is given in equation -1 above; the corresponding flux is:

$$F = \begin{bmatrix} \rho \\ \rho u^2 + p \\ \rho u (e + u^2/2) + up \end{bmatrix} \tag{27}$$

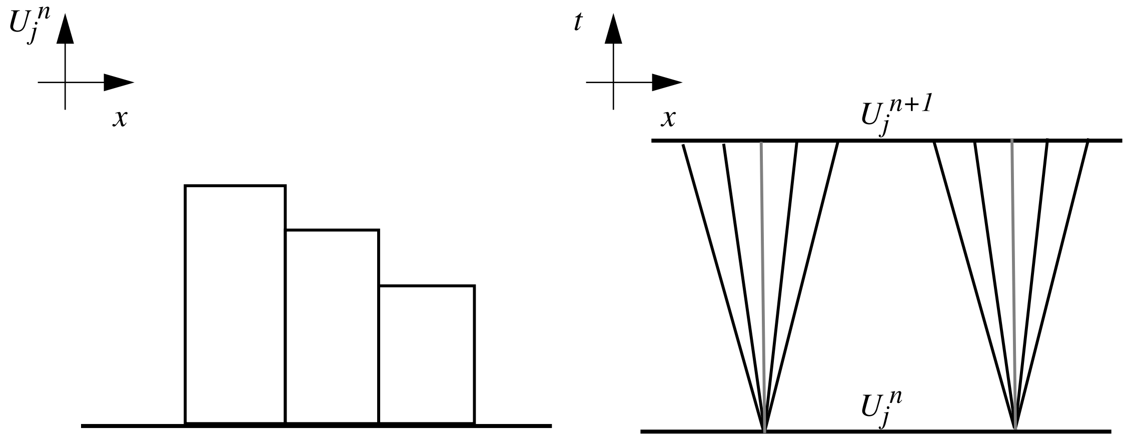
The concern now arises as to what to do about the internal energy *e,* which is not explicitly solved for in the system of three equations. The solution is to solve the system redundantly, adding an extra equation which solves for *e*. This essentially adds an equation of state to this system of equations. For continuum problems, we might want to adopt this approach to put other things, e.g., stresses, into the Riemann problem.

(3) **Make the approximation slightly nonlinear.** So far, we've talked about computing *R* at a single point to obtain $R((U_L + U_R)/2)$. Adding a small amount of nonlinear information can make the scheme much more robust. (I'm not sure how to expand on this, but it's rather vague)

4.4 Temporal evolution

4.4.1 First-order Godunov

As we've done before, the solution can be updated by first-order Godunov, an upwinding technique which also satisfies the entropy condition by dissipating rarefaction shocks. The solution looks like a series of constant states separated by discrete jumps:



$$U_j^{n+1} = U_j^n + \frac{\Delta t}{\Delta x}\left[F_{j-1/2} - F_{j+1/2}\right] \tag{28}$$

where the flux is given by:

$$F_{j+1/2} = F\left(U_{RP_{j+1/2}}\right) \tag{29}$$

and $U_{RP_{j+1/2}}$ is the solution to the Riemann problem along the ray *x/t = 0* with $(U_L, U_R) = (U_{j-1}^n, U_{j+1}^n)$.

4.4.2 Second-order Godunov

We can also solve this problem with higher-order accuracy by utilizing the predictor/corrector approach of second-order Godunov. The discrete evolution equation for the predictor step is found by taking an upstream-centered Taylor expansion:

$$\begin{aligned} U_j^{n+1} &= U_j^n + \frac{\Delta x}{2}\frac{\partial U}{\partial x}\bigg|_{j\Delta x,\, n\Delta t} + \frac{\Delta t}{2}\frac{\partial U}{\partial t}\bigg|_{j\Delta x,\, n\Delta t} + \ldots \\ &\approx U_j^n + \frac{\partial U}{\partial x}\left[\frac{\Delta x}{2} - A(U)\frac{\Delta t}{2}\right] \\ U_j^{n+1} &= U_j^n + \frac{\partial U}{\partial x}\Delta x\frac{1}{2}\left[I - A(U_j^n)\frac{\Delta t}{\Delta x}\right] \end{aligned} \tag{30}$$

This is the same procedure that we have followed before. However, at this point we need to be a little more creative, because we can not diagonalize the nonlinear operator on the r.h.s. of the equation. Note that the form of the equation looks like a perturbation around the state U_j^n. Recall that we talked about perturbation propagation along characteristics in section xxx.xxx.x, so that we can think about writing the first part of the second term as:

$$\frac{\partial U}{\partial x}\Delta x = \sum \alpha_k r_k \tag{31}$$

Plugging this into equation -2, we can continue:

$$U_j^{n+1} = U_j^n + \frac{1}{2}\left[I - A(U_j^n)\frac{\Delta t}{\Delta x}\right]\sum \alpha_k r_k$$

$$U_j^{n+1} = U_j^n + \sum_k \frac{1}{2}\left[I - \lambda_k \frac{\Delta t}{\Delta x}\right]\alpha_k r_k \tag{32}$$

This indicates that for $\lambda > 0$, signals propagate to the right along characteristics. Exactly as for the linear case, define the limiter as:

$$\Delta U_j = \Delta x \left.\frac{\partial U}{\partial x}\right|_{j\Delta x} \tag{33}$$

and the left, right and center amplitudes as

$$\alpha_{k_j}^L = l_k \cdot (U_j^n - U_{j-1}^n) \tag{34}$$

$$\alpha_{k_j}^C = \frac{1}{2} l_k \cdot (U_{j+1}^n - U_{j-1}^n) \tag{35}$$

$$\alpha_{k_j}^R = l_k \cdot (U_{j+1}^n - U_j^n) \tag{36}$$

and

$$\alpha_{k_j} = \left\{ \begin{matrix} (sign(\alpha_{k_j}^C))\, min(2|\alpha_{k_j}^L|, |\alpha_{k_j}^C|, 2|\alpha_{k_j}^R|) \ldots if \ldots (\alpha_{k_j}^L \alpha_{k_j}^R > 0) \\ 0 \ldots otherwise \end{matrix} \right. \tag{37}$$

Now compute the left and right states at the half step, using the projection operators:

$$U_{j+1/2_L}^{n+1/2} = U_j^n + \frac{1}{2}P_+\left(\left(I - A(U_j^n)\frac{\Delta t}{\Delta x}\right)(\Delta U_j)\right) \tag{38}$$

$$U_{j+1/2_R}^{n+1/2} = U_{j+1}^n - \frac{1}{2}P_-\left(\left(I + A(U_{j+1}^n)\frac{\Delta t}{\Delta x}\right)(\Delta U_{j+1})\right) \tag{39}$$

where the projection operators are defined by:

$$P_{+}(W) = \sum_{k|\lambda_k > 0} [l_k(U_j^n) \cdot W] r_k(U_j^n) \tag{40}$$

$$P_{-}(W) = \sum_{k|\lambda_k < 0} [l_k(U_j^n) \cdot W] r_k(U_j^n) \tag{41}$$

(Needs some more erudition on projection operators.) Now solve the Riemann problem, using the left and right states defined by the equations above. Next compute the fluxes and update using conservative finite differencing. The tricks that we may want to employ are using primitive variables or some other nonlinear change of variables and/or augment the variables to properly compute $U_{j+1/2}^{n+1/2}$