PART II. Incompressible flow

2. Prototype of advection equation.

In two dimensions, our model of the advection equation is

$$\frac{\partial \rho}{\partial t} + (\underset{\sim}{u} \cdot \nabla) \rho = 0 \tag{1}$$

Assume that $\underset{\sim}{u} = (u, v)$, where u and v are positive constants. What are our requirements for a decent scheme? We will want a good first-order method which satisfies the maximum principle and has reasonable stability properties in terms of CFL number. We will then add to this method the minimum number of terms required to make the scheme second order in smooth regions, but must be can be taken out at discontinuities to avoid Gibb's phenomenon.

As usual, we will assume that the mesh spacing is uniform in both dimensions. Now think of ρ_{ij}^n as the integrated average of the advected scalar, e.g., mass, in cell *(i, j)* at time n, i.e.

$$\rho_{ij}^n \approx \frac{1}{\Delta x \Delta y} \int \rho (x, y, t^n) \, dx dy \tag{2}$$

What happens as we move forward in time? The mass in cell *(i, j)* at time $n+1$ results from advection directly from the cell itself at time n, as well as contributions from its nearest neighbors. Recalling that u and v are positive, we can express the mass in cell *(i, j)* at the next time level using piecewise constant interpolation:

$$\rho_{ij}^{n+1} = \frac{1}{\Delta x \Delta y} [A_1 \rho_{ij}^n + A_2 \rho_{i-1,j}^n + A_3 \rho_{i,j-1}^n + A_4 \rho_{i-1,j-1}^n] \tag{3}$$

The areas A_1 to A_4 are all positive and at most equivalent to the cell area $\Delta x \Delta y$ (for the case of purely one-dimensional flow in the x direction). Since we are using the area-weighted average of the values at the old time, we will introduce no new maxima or minima by using this evolution equation. Figure +1 below depicts this process graphically. In this figure, the large region represents the domain at time level $n+1$, with cell *(i, j)* being at the center (cross-hatched region). There are four contributions coming from the cell itself and the adjacent cells at time n, shown by the shaded region. All of the contributions are advected at the speed $\underset{\sim}{u}\Delta t$.

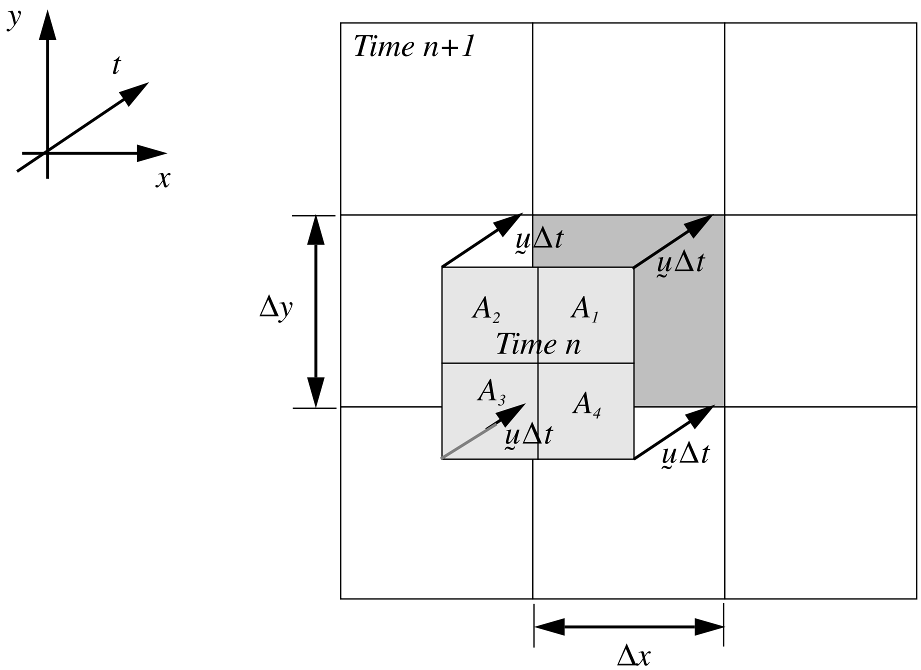

FIGURE 1. Areas through which mass is transported to cell *(i, j)* at time *n+1*.

Next, we'd like to figure out just what the flux of mass through the cell edges are, e.g., the right cell edge $i + \frac{1}{2}$. We note that the velocity components will indicate how much flux is moving through in either the x or y direction. Let's make up the following notation to denote the areas through which relevant flux traverses:

$$B_o = (u\Delta t)\,\Delta y \tag{4}$$

$$B_B = \frac{1}{2}(u\Delta t)\;(v\Delta t) \tag{5}$$

$$B_T = \frac{1}{2}(u\Delta t)\;(v\Delta t) \tag{6}$$

Graphically,

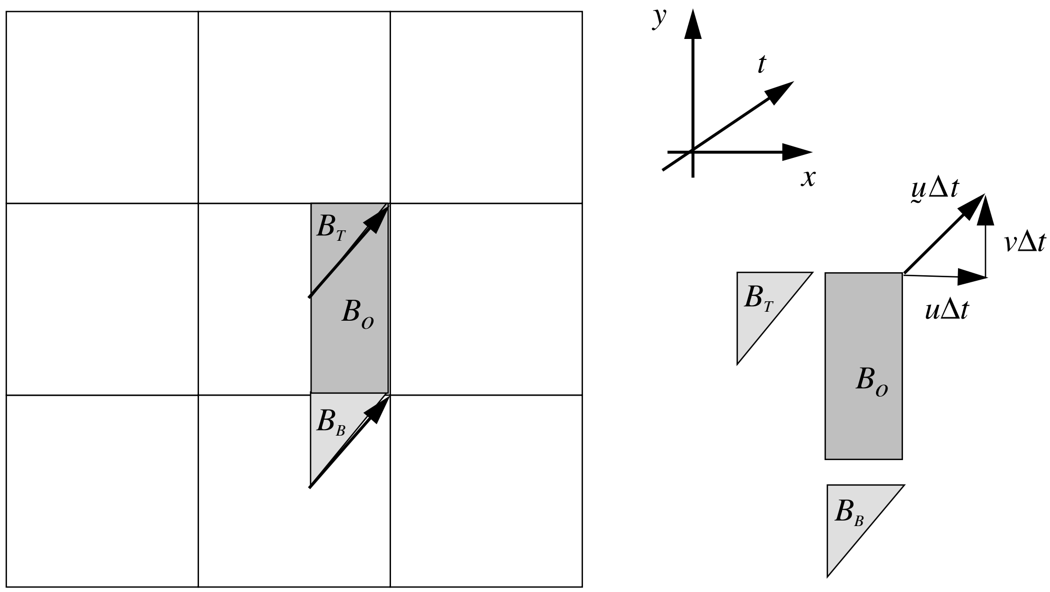

FIGURE 2. Flux which enters or leaves through right cell edge between time *n* and *n+1*

We can then express the mass flux out of edge $i + \frac{1}{2}$, $F_{i+1/2}$, as:

$$F_{i+\frac{1}{2}} = B_o \rho^n_{ij} + B_B \rho^n_{i,j-1} - B_T \rho^n_{ij}$$

$$= (u\Delta t)\, \Delta y \rho^n_{ij} + \frac{1}{2}(u\Delta t)\ (v\Delta t)\ (\rho^n_{i,j-1} - \rho^n_{ij})$$

and introducing the definition for $\rho_{i+1/2,j}$:

$$= (u\Delta t)\, \Delta y \rho_{i+1/2,j}$$

so that

$$\rho_{i+1/2,j} = \rho^n_{ij} + \frac{v\Delta t}{2\Delta y}(\rho^n_{i,j-1} - \rho^n_{ij}) \tag{7}$$

If we perform similar analyses for the upper edge, we can find a similar expression:

$$\rho_{i,j+1/2} = \rho^n_{ij} + \frac{u\Delta t}{2\Delta x}(\rho^n_{i-1,j} - \rho^n_{ij}) \tag{8}$$

ultimately arriving at the following evolution equation:

$$\rho^{n+1}_{ij} = \rho^n_{ij} + \frac{u\Delta t}{2\Delta x}(\rho^n_{i-1/2,j} - \rho^n_{i+1/2,j}) + \frac{v\Delta t}{2\Delta y}(\rho^n_{i,j-1/2} - \rho^n_{i,j+1/2}) \tag{9}$$

This evolution equation is only first-order accurate. Is there something we can do about that? Equation (-2) above is an approximation for

$$\rho^{n+1/2}_{i,j+1/2} = \rho^n_{ij} + \frac{\Delta x}{2}\frac{\partial\rho}{\partial x} + \frac{\Delta t}{2}\frac{\partial\rho}{\partial t}$$

Substituting in from the PDE:

$$\rho^{n+1/2}_{i,j+1/2} = \rho^n_{ij} + \frac{\Delta x}{2}\frac{\partial\rho}{\partial x} - \frac{u\Delta t}{2}\frac{\partial\rho}{\partial x} - \frac{v\Delta t}{2}\frac{\partial\rho}{\partial y}$$

$$= \rho^n_{ij} + \left(\frac{\Delta x}{2} - \frac{u\Delta t}{2}\right)\frac{\partial\rho}{\partial x} - \frac{v\Delta t}{2}\frac{\partial\rho}{\partial y}$$

As before, we'd like to apply a limiter to the term multiplied by $\partial\rho/\partial x$ and in addition, use upwind differencing on the term multiplied by $\partial\rho/\partial y$. Let's give it the old van Leer heave ho:

$$\rho^{n+1/2}_{i,j+1/2} = \rho^n_{ij} + \frac{1}{2}\left(1 - \frac{u\Delta t}{2\Delta x}\right)\Delta_x\rho_{ij} - \frac{\Delta t}{2\Delta y}v\left(\rho^n_{ij} - \rho^n_{i,j-1}\right) \quad (10)$$

where

$$\Delta_x\rho_{ij} = sign\left(\rho_{i+1,j} - \rho_{i-1,j}\right) \cdot$$

$$min\left(2\left|\rho_{i,j} - \rho_{i-1,j}\right|, \frac{1}{2}\left|\rho_{i+1,j} - \rho_{i-1,j}\right|, 2\left|\rho_{i,j} - \rho_{i+1,j}\right|\right) \quad (11)$$

if $(\rho_{i-1,j} - \rho_{ij}) \cdot (\rho_{ij} - \rho_{i-1,j}) > 0$; or is set to zero otherwise.

This scheme is not monotonicity-preserving. While this is not a major drawback from the standpoint of linear systems, it is a problem for nonlinear advection. Two classes of fixes have been proposed: augment the van Leer limiter using B limiters (J. Saltzman); or do the interpolation (advection) in a more appropriate fashion (van Leer, 1984; Bell, Dawson and Shubin, 1989).