A Bibliography of Publications on the Numerical Calculation of $\pi$

Nelson H. F. Beebe  
University of Utah  
Department of Mathematics, 110 LCB  
155 S 1400 E RM 233  
Salt Lake City, UT 84112-0090  
USA  
Tel: +1 801 581 5254  
FAX: +1 801 581 4148  
E-mail: beebe@math.utah.edu, beebe@acm.org, beebe@computer.org (Internet)  
WWW URL: http://www.math.utah.edu/~beebe/

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(sin $\alpha$)/$\alpha$ [128]. 0 [251], 1 [264], 1/$\pi$ [224, 225, 318, 289]. 1/$\pi^2$ [243, 258, 225, 10,000 [57]. 10,000,000 [155]. 16 [233]. 2,000 [39]. 2,576,980,370,000 [252]. $\$24.95 [222]. 29,360,000 [112]. $2H_2$ [262]. $b$ [210]. $C$ [306]. $d$ [306]. $e$ [221, 113, 107, 65, 38, 126, 32, 39, 40, 250, 13, 63]. $e^{-\pi/2} = \nu$ [15]. $\gamma$ [77]. GL$(n, Z)$ [110]. $N$ [129, 163, 96, 110, 154]. $\phi$ [223, 230]. $\pi$ [156, 221, 276, 111, 157, 203, 269, 35, 183, 140, 112, 113, 270, 313, 28, 23, 200, 70, 78, 139, 163, 17, 107, 309, 165, 92, 95, 101, 102, 119, 261, 44, 65, 212, 18, 223, 230, 298, 231, 71, 262, 216, 88, 166, 55, 152, 217, 66, 38, 213, 37, 24, 133, 299, 4, 272, 26, 21, 293, 128, 5, 9, 10, 285, 180, 232, 143, 148, 233, 115, 187, 116, 246, 122, 126, 247, 188, 93, 117, 286, 167, 181, 72, 27, 134, 182, 22, 129, 105, 135, 32, 39, 84, 234, 68, 97, 47, 29, 196, 168]. $\pi$ [207, 57, 48, 235, 7, 214, 149, 14, 202, 40, 76, 19, 6, 58, 77, 274, 69, 11, 12, 36, 250, 175, 252, 310, 94, 62, 123, 30, 176, 220, 131, 16, 13, 145, 169, 302, 155, 53, 192, 63, 8, 170]. $\pi, e$ [87, 106]. $\pi/12$ [31]. $\pi/4$ [46]. $\pi/8$ [31]. $\pi = 2 \sum \arccot f_{2k+1}$ [79]. $\pi^2$ [260, 279, 125, 48]. $\pi^4$ [104]. $\pi \coth \pi$ [236]. $q$ [246]. $\sqrt{2}$ [61, 64]. $\sqrt{2} + \sqrt{2}$ [250]. $\sum 1/k^2 = \pi^2/6$ [67]. $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$ [54]. $\sum_{k=1}^{\infty} = \pi^2/6$ [73]. $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$ [108]. $\sqrt{2}$ [87]. $Z$ [110]. $\zeta(2) = \frac{\pi^2}{6}$ [288].
BBP-Type [259, 264, 240, 241, 242, 253, 254, 255, 256, 257, 278].
Benford [268]. Berggren [325]. Berkeley [198]. Best [28, 326]. better
[221], between [118]. Beyond [292]. bilateral [262]. billion [160, 119, 175].
billionth [162]. Binary [240, 241, 253, 254, 256, 208, 162, 163, 61, 64].
both [290]. Bouncing [315]. Boy [304]. Brent [86, 100]. Bresenham [202].

C [77]. calculate [190]. calculated [39]. calculates [312]. Calculation
[200, 18, 9, 10, 148, 93, 84, 207, 219, 58, 175, 94, 145, 311, 88, 4, 5, 117, 22, 68,
Catalan-Type [250]. Catalan’s [260]. catalogue [313]. Celebrating [301].
Celebration [327]. Central [143]. Century [23, 301, 133, 211]. certain
Choong [77]. Christmas [263]. Chronology [41, 42, 43]. cifre [72]. Circle
classical [120]. Classroom [46, 84, 73, 48, 67]. cluster [310]. Colin [222].
Communicating [228]. Comp [77]. comparative [324]. Compendium
[259, 278]. Complex [173]. complexity [75, 244, 319]. Comprising [3].
Computation [112, 215, 260, 309, 24, 47, 76, 19, 11, 12, 251, 310, 275, 159,
271, 279, 120, 109, 178, 122, 154, 7, 202, 155]. Computational [303, 204, 319].
[28, 309, 286, 287, 179, 300, 315]. Concerning [65]. Conclusion [37].
Conjecture [65, 50]. Conjectured [224, 269, 261]. considerations [109].
[241, 255, 113, 161, 259, 205, 189, 118, 159, 278, 120, 63]. Construction
Contributions [3]. convenient [7]. convergence [273]. Convergent
[112, 92]. Converging [102]. Correct [34, 157]. Correspondence [62].
[95, 101]. CUDA [286].

D [77]. Day [281, 291, 294, 239]. Daykin [77]. debate [221, 222]. Decimal
[112, 293, 93, 218, 40, 53, 38, 10, 117, 72, 39, 154, 175, 252, 155].
decimale [72]. décimales [60]. Decimals [309, 3, 58, 94, 60, 233]. Degree
Department [319]. dependence [50]. Dependent [110]. Derivation


7


quadratic [216]. Quadratically [102]. Quadrillionth [287, 251, 310].
[112]. quelques [2]. Quest [158, 231]. Questions [28, 17, 18, 24, 26, 19, 16].

R [77]. Rabbinical [176]. Ramanujan
Ramanujan-like [243, 258, 262]. Ramanujan-type [318]. Random
[195, 199, 280, 153, 217, 293, 45, 198, 282, 283, 226]. Randomness
[303, 218, 57, 220, 245, 38, 197]. rapid [159]. rapidly [92]. rare [217].
Rathbone [77]. Rational [65, 71, 91, 77, 318]. Ratios [31]. Real
Reconstruction [272]. Record [266, 290, 312]. Rectification [3].
Refutation [226]. Regular [77]. Relation
[256, 15, 184, 93, 127, 194, 114, 177, 144]. Relations
[17, 110, 264, 27, 136, 137, 118, 98, 121]. Remark [82]. remarkable [2].
[224, 246]. Reproducibility [303]. Researcher [198]. Researchers [300].
Resolution [239]. Results [113, 16, 118]. Review [222, 305]. Revisited

S3071 [287]. Salamin [134, 100]. Satisfy [152]. Science [303, 321, 319].
Sciences [325]. Scientific [215]. Search [150]. Searching [212]. seeming
[197]. September [327, 320]. Sequence [187, 97, 249]. serial [61, 64]. Series
[24, 224, 225, 46, 267, 243, 216, 318, 129, 97, 207, 7, 131, 289, 202]. Shanks
[34]. Short [103]. Simple [156, 311, 26, 104, 46, 73]. simplified [100].
Slice [206]. Some [203, 309, 232, 47, 16, 261, 2]. source [164, 185, 209].
[321]. Stands [290]. statistica [72]. Statistical [152, 39, 57, 246, 72, 60].
Statistically [293]. statistics [317]. statistique [60]. Steinhaus [51]. Still
[291, 281]. String [258, 243]. Students [295]. Studies [325]. Study
[303, 57, 320, 220, 267, 324, 60, 126]. stumbled [300]. such [226]. suggested
[123]. Sulla [72]. Summing [267]. Sums [141, 114]. Supercomputer
[287, 290]. Supercomputing [322]. Survey [11, 12]. Surveys [325].

Table [77]. tackled [197]. Takebe [168]. Talking [172]. tangent [131].
Taust [294]. Teaching [128]. Techniques [194]. ten [275]. Terms
REFERENCES


Xeon [310].

y-cruncher [316]. Year [150]. yields [129]. Youqin [169].

Zach [302]. Zahl [30, 8]. Zero [256, 264, 75, 244]. zero-finding [75, 244]. Zhao [169].

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In this famous paper, Lambert proved that \( \pi \) is irrational. See [167] for further remarks, a simplification of the proof, and references to earlier papers that discuss Lambert’s proof.


[8] Carl Louis Ferdinand von Lindemann. Über die Zahl \( \pi \). (German) [On the number \( \pi \)]. *Mathematische Annalen*, 20(??):213–225, ???? 1882. CODEN MAANAE. ISSN 0025-5831 (print), 1432-1807 (electronic). In this famous paper, von Lindemann proved that \( \pi \) is transcendental, showing that it is impossible to square the circle by compass and straightedge, a problem dating back before 400 BCE in Greece.


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**Shanks:1962:CD**

[58] Daniel Shanks and John W. Wrench, Jr. Calculation of $\pi$ to 100,000 decimals. *Mathematics of Computation*, 16(77):76–99, January 1962. CODEN MCMPAF. ISSN 0025-5718 (print), 1088-6842 (electronic). URL http://www.jstor.org/stable/2003813. A note added in proof says: “J. M. Gerard of IBM United Kingdom Limited, who was then unaware of the computation described above, computed $\pi$ to 20,000 D on the 7090 in the London Data Centre on July 31, 1961. His program used Machin’s formula, (1) $\pi = 16 \arctan(1/5) - 4 \arctan(1/239)$, and required 39 minutes running time. His result agrees with ours to that number of decimals.”.

**Smith:1966:CP**


**Esmenjaud-Bonnardel:1965:ESD**


**Good:1967:GST**


**Tee:1967:CP**


**Yarbrough:1967:PCC**

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Kanada:1983:CDP


Tamura:1983:CDB


Borwein:1984:CHO


Borwein:1984:EAO


Newman:1984:SAS


Haastad:1985:PTA

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theoretical computer science of the Gesellschaft für Informatik (G.I.) and the special interest group for applied mathematics of the Association française des sciences et techniques de l’information, de l’organisation et des systèmes (AFCET)."


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Parks:1986:NOI


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Ferguson:1987:NIA


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Barrow:1996:PSC


Dodge:1996:DSA


Dodge:1996:NRN


Plouffe:1996:CTD


Wei:1996:CDD

[156] Victor Adamchik and Stan Wagon. Notes: A simple formula for $\pi$. *American Mathematical Monthly*, 104(9):852–855, November 1997. CODEN AMMYAE. ISSN 0002-9890 (print), 1930-0972 (electronic). URL http://www.maa.org/pubs/monthly_nov97_toc.html. The authors employ Mathematica to extend earlier work of Bailey, Borwein [119], and Plouffe, [159], done in 1995, but only just published, that discovered an amazing formula for $\pi$ as is a power series in $16^{-k}$, enabling any base-16 digit of $\pi$ to be computed without knowledge of any prior digits. In this paper, Mathematica is used to find several simpler formulas having powers of $4^{-k}$. They also note that it has been proven that their methods cannot be used to exhibit similar formulas in powers of $10^{-k}$.


Bellard:1997:BBD


Bellard:1997:NFC

[163] Fabrice Bellard. A new formula to compute the $n$-th binary digit of $\pi$. This formula is claimed in [251] to be somewhat faster to compute than the BBP formula., 1997. URL http://bellard.org/pi/pi_bin.pdf.

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Blatner:1997:JP


Delahaye:1997:FNc


Laczkovich:1997:LPI

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Percival:2000:PDE


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Bailey:2001:PIR

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Chong:2008:EQ


Guillera:2008:EPS


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Kaneko:2010:NNP


Osler:2010:LBF


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[260] David H. Bailey, Jonathan M. Borwein, Andrew Mattingly, and Glenn Wightwick. The computation of previously inaccessible digits of $\pi^2$ and Catalan’s constant. Report, Lawrence Berkeley National Laboratory; Centre for Computer Assisted Research Mathematics and its Applications (CARMA), University of Newcastle; IBM Australia, Berkeley, CA, USA; Callaghan, NSW 2308, Australia; St. Leonards, NSW 2065, Australia; Pyrmont, NSW 2009, Australia, April 11, 2011. 18 pp. URL http://crd.lbl.gov/~dhbailey/dhbpapers/bbp-bluegene.pdf.
Borwein:2011:PSE

[261] D. Borwein and Jonathan M. Borwein. Proof of some experimentally conjectured formulas for $\pi$. Preprint, Department of Mathematics, University of Western Ontario and Centre for Computer-assisted Research Mathematics and its Applications (CARMA), School of Mathematical and Physical Sciences, University of Newcastle, London, ON, Canada and Callaghan, NSW 2308, Australia, December 4, 2011.

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Yee:2011:LC


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[281] David H. Bailey and Jonathan Borwein. Pi day is upon us again and we still do not know if pi is normal. Report, Lawrence Berkeley National
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Gourevitch:2013:W


Karrels:2013:CDC


Karrels:2013:SCQ

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[290] Alexander Yee and Shiguro Kondo. It stands at 10 trillion digits of pi... world record for both desktop and supercomputer!!! Web site, April 15, 2013. URL http://www.numberworld.org/y-cruncher/. This site also contains a table of digit records from 2009 to 2013 for various mathematical constants. The $\pi$ record of 10,000,000,000,050 decimal digits was reached on 17 October 2011 after 371 days of computation, and 45 hours of verification.


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