The mathfont Package in Action: Two Mathematical Snippets Rendered in Times New Roman

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Mathematicians usually define e in one of two ways: as the horizontal asymptote of a certain function or as the limit of an infinite series. Specifically, it's most common to see e defined as either

$$e = \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$$

or

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}.$$

The first definition is simpler in that involves a limit of a single expression, not a limit of partial sums, but in practice, the second tends to be more tractable. The power series expression of e^x is given by

$$\sum_{n=0}^{\infty} \frac{x^n}{n!},$$

and the relationship between this expression and the series definition is much more apparent than it is for the first limit. This relationship arises in a variety of different mathematical contexts, for example the famous Euler's formula $e^{i\theta} = \cos\theta + i \sin\theta$ or the related definition of the characteristic function for a random variable X:

$$\phi_X(t) = \mathbb{E}\left(e^{iX}\right).$$

Expanding e^{iX} as a power series gives an expression for ϕ_X that we can differentiate term by term.

A smooth manifold consists of a topological space M equipped with a smooth maximal atlas $\{\phi_i\}$. The maps $\phi_i \colon U_i \longrightarrow \mathbb{R}$ technically aren't themselves differentiable, but their compositions $\phi_i \circ \phi_j^{-1}$ are diffeomorphisms on subsets of $\mathbb{R}^n$. If we have a map $f \colon M \longrightarrow N$ between manifolds, this structure allows us to talk about differentiability of f. Specifically, we say that f is smooth if for any i and j, the composition

$$\psi_j \circ f \circ \phi_i^{-1}$$

is itself smooth, where $\{\psi_i\}$ is a smooth atlas for N. Differentiating f produces the associated tangent map Df. The function Df maps the tangent bundle TM to the tangent bundle TN and is linear when restricted to individual tangent spaces T_pM. If M can be written as a product $M_1 \times M_2$, we can consider the partial tangent maps $\partial_1 f$ and $\partial_2 f$ by considering the compositions $f \circ \imath_1$ and $f \circ \imath_2$, where $\imath_1$ and $\imath_2$ are inclusion maps with respect to a particular point. Combining both maps, we have the equation

$$Df(u, v) = \partial_1 f(u) + \partial_2 f(v),$$

and this relationship can be thought of as an adaption of the standard product rule.