Math 5710, Supplement 1

More about mechanical equilibrium

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The configurations of a mechanical system with a finite number (n) of degrees of freedom is often describable by specifying n generalized coordinates $x_1, x_2, \ldots x_n$, i.e. the column vector x. If the mechanical system is acted upon by an external force field which is conservative (such as gravity), it generally has a potential energy function $V(x)$ which may have a minimum value.

The equilibrium configuration of the system, if one exists, is a value $x = x^{eq}$ which minimizes $V(x)$:

$$V(x^{eq}) = \min_x V(x).$$

Typically there is only one, although there may be more. More generally, relative minima provide relative equilibria.

Generalized forces act on the system. They are defined by the vector function

$$F(x) = -\nabla V(x),$$

i.e.

$$F_i = -\frac{\partial}{\partial x_i} V(x), \qquad i = 1, 2, \ldots, n$$

Thus $F = 0$ at $x = x^{eq}$.

Generally, V consists of a part due to the interactions among the elements of the mechanical system plus linear terms due to the external force field:

$$V = V_{int} - \sum_i f_i x_i.$$

If the interactions are all purely elastic and the x's are displacements of certain material points from some reference positions, then for V we get a quadratic form plus linear form:

$$V(x) = \sum_{i,j} b_{ij} x_i x_j - \sum_i f_i x_i \tag{1}$$

for some numbers f_i and some positive definite symmetric matrix $B = (b_{ij})$. The matrix $2B$ is then called the stiffness matrix.

In more realistic cases, the interactions are not purely elastic; then V takes on some more general form. Nevertheless, near $x = x^{eq}$, one can do a Taylor series expansion of the function V, in the form

$$V(x) = \sum_{i,j} b_{ij}(x_i - x_i^{eq})(x_j - x_j^{eq}) + \text{ higher order terms},$$

where B is symmetric positive definite. So if $x - x^{eq}$ is small, i.e. if the system is close to equilibrium, we can ignore the higher order terms and obtain something like (1) above.

Consider, as example, the spring system in Fig. 1.7, p. 40. The elastic energy of a purely elastic spring is always given by $\frac{1}{2}ce^2$, where c is the spring constant and e is the elongation. Therefore in this case,

$$\begin{aligned} V &= V_{elastic} + V_{grav}, && (2)\\ V_{elastic} &= \frac{1}{2}\sum_{i=1}^{4} c_i (e_i)^2, && (3)\\ V_{grav} &= -\sum_{j=1}^{3} m_j g x_j, && (4)\\ & && (5) \end{aligned}$$

where the c's, m's, and e's are as given in the book and g is acceleration due to gravity. In particular,

$$e = Ax,$$

where A is the 4×3 matrix given in the text. The matrix A simply describes the geometry of the layout; it gives the relationship between the elongations and displacements. On the other hand the c's and m's are physical constants.

Thus

$$V = \frac{1}{2}\sum_{i=1}^{4} c_i((Ax)_i)^2 - g\sum_{j=1}^{3} m_j x_j. \tag{6}$$

The first term on the right is a quadratic form in the vector x with some new matrix B. In fact, since

$$((Ax)_i)^2 = (\sum_{k=1}^{4} a_{ik} x_k)(\sum_{\ell=1}^{4} a_{i\ell} x_\ell),$$

it can be written as

$$\frac{1}{2}\sum_{k,\ell}(\sum_{i} c_i a_{ik} a_{i\ell}) x_k x_\ell,$$

so that B has elements

$$b_{k\ell} = \frac{1}{2}\sum_{i=1}^{4} c_i a_{ik} a_{i\ell}. \tag{7}$$

Check that this means $B = \frac{1}{2}A^T C A$, for what matrix C?

Thus (6) becomes

$$V(x) = \sum_{i,j=1}^{3} b_{ij} x_i x_j - g \sum_{k=1}^{3} m_k x_k.$$

The matrix B is symmetric and positive definite. The symmetry follows just from (7); it is seen to be unchanged when k and ℓ are reversed. As for the positive definiteness, we just show that the first term in (6) is positive except when $x = 0$. Since the numbers c_i are all positive, that expression will be positive except when $Ax = 0$. But you can check easily from the form of A that $Ax = 0$ implies $x = 0$. We are done.

The generalized forces associated with each coordinate x_j, i.e. acting on each mass m_j, are

$$F_j = -\frac{\partial}{\partial x_j} V(x) = -2\sum_{k} b_{jk} x_k + g m_j, \quad j = 1, 2, 3.$$

The last term here is simply the weight of the j-th mass. The summation on the right represents the sum of the two elastic forces acting on the mass m_j by the two attached springs.