Pattern formation in gradient systems*

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Abstract

Stable and metastable patterned solutions of nonlinear evolution equations of gradient type are discussed. Examples include classes of higher order parabolic PDE's and integrodifferential equations. A governing theme is that patterns can arise as a result of a competition between opposing influences such as destabilizing and stabilizing mechanisms. The discussion is within the context of a general framework although in the case of conserved evolutions, most attention is given to the Cahn-Hilliard equation and metastable patterns. The focus is on rigorous results; however some important formal stability and modulational theories are also reviewed.

1 Introduction

Spatial patterns observed in natural phenomena are often the product, in some perhaps loose sense, of competition between a destabilizing and a stabilizing mechanism. The main aim of this chapter is to explore the mathematical implications of this type of competition in the context of systems that have a Lyapunov functional. Within this restricted approach to patterns, we shall try to be general, in an attempt to clarify the essential mathematical ingredients in their formation. In most of the chapter our setting, in fact, will be evolution laws, for functions $u(x,t)$, of the form

$$u_t = Au - \rho Bu - f(u), \tag{1}$$

or this equation's conserved analog, where A and B are negative (not necessarily differential) self-adjoint operators on a space of functions of x, and f is a function such that $uf(u)$ grows rapidly enough as $|u|\to\infty$. The first two terms on the right of (1) represent the stabilizing and destabilizing influences, respectively, while the last term serves to limit the growth of patterned solutions. The number ρ will be our control parameter; stable patterns arise when it surpasses a critical value.

There has been a vast number of mathematical investigations into patterned solutions of nonlinear partial differential equation models in the sciences; the reader is referred for example to [31] for an excellent comprehensive review. Our focus here is much more restrictive, in that we emphasize the role of mathematical rigor, and of course limit our attention to gradient systems.

These restrictions exclude detailed treatment of a number of well-developed multiscale theories directed to the stability and other properties of patterns. They employ the tools of linear and weakly nonlinear stability analyses, amplitude equations, and phase evolution equations. We describe the ideas underlying such theories briefly in secs. 2.1, 2.3, and especially 5. The prototypical

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applications of those methods have been to fluid dynamical instabilities (see for example the exposition in [38]), but the list extends far beyond that discipline. We are also excluding the extensive modeling, simulations, and asymptotic analyses that have been done for (nongradient) patterns in excitable media. One very good survey of this field is [64]. The study of patterns arising through bifurcation has been prevalent in the modeling literature since the seminal paper by Turing [82]. Again, we restrict our attention here to bifurcating solutions within our stated framework (Sec. 3.2).

In our context, patterns will be defined as stable stationary spatially nonconstant solutions of evolution equations. The concept of stability can take various forms, all related to the expected behavior of other solutions of the evolution problem starting near the one in question. We shall speak of solutions being stable either in the sense of (1) their being minimizers of the Lyapunov functional, or (2) satisfying the formal requirements of some linear or weakly nonlinear stability theory.

In the case of (1), a minimizer ϕ has the property that no solution trajectory can leave it; i.e. there is no solution $w(t)$ defined for all t, such that $w(-\infty) = \phi$. In particular, there is no unstable manifold. This is because the Lyapunov functional would have to decrease on that trajectory, which would contradict ϕ being a minimizer. In the case of linear stability criteria (2), one replaces the evolution equation by its linearization about the given stationary solution ϕ, and again asks whether there can exist any trajectory leaving ϕ. The search is limited to solutions behaving exponentially in t, i.e. involving a factor $e^{\sigma t}$. Stability occurs if all such coefficients σ satisfy $\Re\sigma < 0$; if this is true except that some satisfy $\Re\sigma = 0$, then further analysis, if possible, is called for. Very often, one or both of these notions of stability imply that for some Banach space X and any $\epsilon > 0$ there is a number $\delta(\epsilon) > 0$ such that $\|u(t) - \phi\|_X < \epsilon$ for $t > 0$ whenever $u(t)$ is a solution with $\|u(0) - \phi\|_X < \delta$. See, for example, Comment 1 following Thm. 1.

The simplest scenario in which patterns can be studied is that of equations which admit complex exponential solutions, and Section 2 is concerned solely with this case. A general linear stability criterion was developed by Newton and Keller for this situation; it is reviewed in Section 2.1. It applies to problems of greater generality than those within the stated theme of this chapter.

Criterion (1) is the focus of the material in Section 2.2, which is based on [43]; in fact, we deal exclusively with conditions for the existence of noncontant global minimizers for our general class of gradient flows, together with many properties of those (sinusoidal) minimizers. The problem is posed in the context of spatially periodic solutions with given wavelength λ, although our intention is to discover properties of those solutions which change little with λ when it is large. At the end we return to the linear stability criterion, which can be spelled out explicitly for our problems. We show that stability in this sense, as expected, encompasses a larger set of solutions than just global minimizers.

We pass in Section 3 to consider problems which do not have complex exponential solutions. Less can be said in this case: since the solutions are not known explicitly, linear stability analyses would be far more difficult. However, necessary as well as sufficient conditions can be given for the existence of globally minimizing patterns. These two conditions just don't coincide, as they do in the cases considered in Section 2.

The control parameter ρ is a measure of the relative strength of the destabilizing mechanism, compared with that of the stabilizing one. We are able to provide (as in Sec. 2) estimates for the amplitude of the patterns as a function of ρ (they generally grow unboundedly as $\rho\to\infty$), but rather little about their dependence on λ.

At the end of this section, we indicate the relevance of our theory to the Swift-Hohenberg and related equations, as well as to activator-inhibitor models, which are prevalent in biological modeling.

Up to this point, the gradient flows will have been simple nonconserving ones with respect to the given Lyapunov (energy) functional. Sec. 4 is devoted to those for a conserved quantity. This class includes the Cahn-Hilliard equation. As far as the existence of minimizers is concerned, much of the previous theory still applies. But an important new phenomenon now arises: that of metastable patterns. When the energy does not admit interesting globally minimizing patterns, it still may be the case that the dynamics drives solutions to a neighborhood of patterned states where they remain temporarily, but for a long time. This is due to a different kind of competition, namely that between the system's desire to minimize the energy, which happens best at long wavelengths, and the kinetics of diffusion, which accomplishes patterning more quickly when the wavelength is small. This phenomenon leads to interesting mathematical issues. Some recent developments in this connection in higher dimensions, particularly due to Maier and Wanner with coworkers, are reviewed, together with "slow motion" results of Alikakos, Bates, Fusco, and others.

This chapter does not attempt a complete review of the subject, nor does it attempt to assign priority to the ideas or results described.

2 Models with complex-valued sinusoidal minimizers

There exist quite a number of nonlinear PDE's arising in mathematical physics that admit complex-valued stationary or traveling wave sinusoidal solutions.

It is generally possible to pursue a linear stability analysis of such solutions. This has been done many times in the past for particular cases. The most complete results in this connection are probably those of Newton and Keller ([69, 70]), who showed that the linear stability criterion can in most cases be reduced to an algebraic condition. We begin by quickly reviewing this in Section 2.1.

In the following Section 2.2, we specialize to gradient flows for an energy functional admitting complex-valued functions, and ask for minimizers of the energy. They are guaranteed to be stable solutions of the corresponding evolution problem. We prove a threshold criterion for their existence, in terms of the "destabilization" parameter ρ, and also derive many properties of these solutions, such as their asymptotic structure (wavelength and amplitude) for large values of ρ. At the end (Sec. 2.3), we return to the linear stability criterion in this special context.

2.1 The Newton-Keller criterion for linear stability

This criterion applies to a wide class of problems with sinusoidal traveling wave solutions. The basic focus of these authors was on systems of differential equations of the form

$$\mathcal{F}(i\partial_t, -i\partial_x, |u|^2)u = 0, \tag{2}$$

for a matrix $\mathcal{F}$ of differential operators. There exists a solution of the form

$$u_0(x,t) = aRe^{i(kx-\omega t)}, \tag{3}$$

provided that the real parameters $a,\ k,\ \omega$ satisfy

$$\det \mathcal{F}(\omega, k, a^2) = 0 \tag{4}$$

and the unit vector R is a right nullvector of the matrix $\mathcal{F}(\omega, k, a^2)$.

A linear stability analysis can be performed for solutions u_0. The analysis is for solutions defined on the entire real line. It is found to be convenient, and no restriction, to write the perturbed solutions in the form

$$u(x,t;\epsilon) = u_0(x,t) + \epsilon e^{i(kx-\omega t)}\phi(x,t) + o(\epsilon). \tag{5}$$

This expression, when substituted into (2) and the higher order (in ϵ) terms disregarded, yields a linear system for ϕ.

It turns out that this linear system has solutions of the form

$$\phi(x,t) = C_1 e^{ilx+\sigma t} + C_2 e^{-ilx+\bar{\sigma}t} \tag{6}$$

for some constant vectors C_1 and C_2, if the numbers l (real) and σ satisfy the solvability condition that the following matrix have zero determinant:

$$\begin{pmatrix} \mathcal{F}(\omega+i\sigma, k+l, a^2) + a^2[\mathcal{F}_{a^2}(\omega,k,a^2)R]\bar{R} & a^2[\mathcal{F}_{a^2}(\omega,k,a^2)R]R \\ a^2[\bar{\mathcal{F}}_{a^2}(\omega,k,a^2)\bar{R}]\bar{R} & \bar{\mathcal{F}}(\omega-i\sigma,k-l,a^2) + a^2[\bar{\mathcal{F}}_{a^2}(\omega,k,a^2)\bar{R}]R \end{pmatrix} \tag{7}$$

If there is such a pair (l,σ) with $\Re\sigma > 0$, then the basic wave u_0 will be unstable according to this criterion. For stability, the criterion is that $\Re\sigma \le 0$ in all cases, and also that every solution of the linear equation for ϕ be expressible as a combination of solutions of the form (6).

Outside of this theory, there are important weakly nonlinear instabilities operating on long time and space scales; they will be reviewed in Sec. 5.

The stated criterion was applied in [69, 70] to various examples, including the nonlinear Schrödinger, Klein-Gordon equations and coupled systems, the Zakharov system for Languir waves in plasmas, and the envelop equation for deep water waves. It yields the classical Eckhaus instabilities in the case of the simple Ginzburg-Landau equation

$$u_t = u_{xx} + u(1-|u|^2).$$

The given method actually applies to a broader class of problems that those stated above. We shall see that the evolutions considered next fall into this broader category.

2.2 Gradient flows: the nonconserved case

This section is based on [43].

The basic ingredients of the evolution equations considered in this section are linear self-adjoint operators A and B and a real function $F(|u|^2)$ growing rapidly enough at ∞. We shall consider gradient flows

$$\frac{\partial u}{\partial t} = -\frac{\delta E[u]}{\delta u} \tag{8}$$

for the energy functional

$$E[u] = -\frac{1}{2}\langle Au,u\rangle + \frac{\rho}{2}\langle Bu,u\rangle + \frac{1}{\lambda}\int_0^\lambda F(|u|^2)dx, \tag{9}$$

which is therefore a Lyapunov functional for the evolution. The flow takes the form (12) below. The roles of A and B are those of a stabilizing and destabilizing mechanism, respectively. The function F serves to limit the growth of any developing patterned mode, thus producing stable finite-amplitude states.

More specifically, the closed self-adjoint operators A and B are negative and densely defined on the L^2 space of complex-valued λ-periodic functions of x. Space is one-dimensional in this section, but in later ones we extend to higher dimensions and at the same time restrict u to be real. We use the scalar product

$$\langle u,v\rangle = \frac{1}{\lambda}\int_0^\lambda u\bar{v}dx.$$

Important examples are

(a) $A=-\partial_x^4, \quad B=\partial_x^2$;

(b) $A=\partial_x^2, \quad B=G*u-u$, where $G*u$ means convolution with a continuous function G in $L^1(R)$, where $G\geq 0$ and $\int_{-\infty}^{\infty}G(x)\,dx=1$.

The wavelength λ is a variable parameter, which can take on any positive value. Thus A and B implicitly depend on λ, being operators on a space which does. Generally, we think of λ as being large; then the *minimal* wavelength of our solutions will typically be much smaller than λ.

Further assumptions on A and B are:

A1. Smooth symbols $\hat{A}(k)$ and $\hat{B}(k)$, defined for all real k, independent of λ, exist such that when e^{ikx} is λ-periodic, $\hat{A}(k)=e^{-ikx}A[e^{ikx}]$, and the same for $\hat{B}$. These functions are real and even.

A2.

$$\lim_{k\to\infty}\frac{\hat{A}(k)}{\hat{B}(k)}=\infty, \qquad \limsup_{k\to\infty}\hat{B}(k)<0. \tag{10}$$

A3. The nullspaces of A and B are the set of constant functions. It follows that $\hat{A}(k)$ and $\hat{B}(k)$ are strictly negative for $k>0$.

We have this Fourier representation: if $u=\sum_m u_m e^{i2\pi mx/\lambda}$, then $Au=\sum_m u_m\hat{A}(2\pi m/\lambda)e^{i2\pi mx/\lambda}$, and the domain of A is the set of functions u such that this series converges in L^2. A similar statement holds for B. Note that **A2** (10) implies the inclusion of domains $\mathcal{D}(A)\subset\mathcal{D}(B)$.

In the example (a) above, $\hat{A}=-k^4, \quad \hat{B}=-k^2$. In (b), $\hat{A}=-k^2$ and $\hat{B}=\hat{G}(k)-1$, where $\hat{G}(k)=\int_{-\infty}^{\infty}G(y)e^{-iky}dy$.

Concerning the function $F(w)$ in (9), we assume it is real and differentiable for $w\geq 0$, with a strict minimum of 0 at $w=u_0^2\geq 0$, such that

$$\lim_{w\to\infty}F'(w)=\infty. \tag{11}$$

We note that $F(w)=(u_0^2-w)^2$ $(u_0>0)$ and $F(w)=\alpha w+w^2, \ \ \alpha\geq 0$, satisfy these assumptions. In the former case, when u is real $F(u^2)$ is a double–well function of u with equal depth wells at $u=\pm u_0$; in the latter case $u_0=0$ and it is a single–well function.

The L_2-gradient flow for (9) is

$$u_t=Au-\rho Bu-f(u), \tag{12}$$

where $f(u)=2uF'(|u|^2)$. We assume this has a global solution $u(t)\in\mathcal{D}(A)\cap L^\infty, \ t\geq 0$, given any initial condition in that space.

Our main goal will be to investigate the possibility of *stable* stationary nonconstant solutions of (12). In fact, we deal mainly with global minimizers of E.

We shall call a minimizer ϕ *nontrivial* if $\phi\neq$ const. In view of **A3** and $F\geq 0$, a sufficient condition for this is that $E[\phi]<0$, since constants have $E\geq 0$.

2.2.1 The minimizers and their properties

We define

$$M(\rho, k) = \hat{A}(k) - \rho\hat{B}(k), \tag{13}$$

an increasing function of ρ, and

$$\rho^*(\lambda) = \inf\{\rho : M(\rho, k_m) > 2F'(u_0^2) \equiv \gamma \text{ for some } m\}, \tag{14}$$

where

$$k_m = \frac{2\pi m}{\lambda}, \quad m = 1, 2, \ldots. \tag{15}$$

We have $0 < \rho^*(\lambda) < \infty$, for if $\rho^*(\lambda) = 0$, then $\hat{A}(k_m) = 0$ for some m, contradicting **A3**.

If $u_0 > 0$, then $\gamma = 0$ in (14). Note the following alternate characterization of ρ^* in that case:

$$\rho^*(\lambda) = \inf_m \frac{\hat{A}(k_m)}{\hat{B}(k_m)} \quad \text{when } u_0 > 0. \tag{16}$$

In fact, $M(\rho, k_m) = -\hat{B}(k_m)[\rho - \frac{\hat{A}(k_m)}{\hat{B}(k_m)}]$, and this is positive for some value of m exactly when ρ is greater than ρ^* as defined by (16).

Below, we speak of functions ae^{ikx}. The solution set of (12) and the set of minimizers of E are invariant under rotation, i.e. multiplication by a constant $e^{i\theta}$. Therefore we may, and shall, always assume that a is real and nonnegative. In this sense, our minimizers are really equivalence classes.

The following theorem provides existence and nonexistence results for nontrivial minimizers, as well as several properties enjoyed by them. See the explanatory comments after the statement of the theorem.

Theorem 1 *(a) For each $\lambda > 0$, $\rho > \rho^*(\lambda)$, there exists a nontrivial global minimizer of E of the form ae^{ikx} for some $a = a(\rho, \lambda) > u_0$, $k = k(\rho, \lambda) > 0$. If $u_0 > 0$, this is also true for $\rho = \rho^*$; then $a(\rho^*(\lambda), \lambda) = u_0$.*

(b) For $\rho < \rho^*(\lambda)$, the only global minimizers are constants with $|u| = u_0$.

(c) The functions a and k satisfy

$$\lim_{\rho\to\infty} a(\rho, \lambda) = \infty, \qquad \lim_{\rho\downarrow\rho^*(\lambda)} a(\rho, \lambda) = u_0. \tag{17}$$

(d) If

$$\inf \hat{B}(k) < \hat{B}(k) \quad \textit{for each} \quad k \tag{18}$$

(the former could be $-\infty$), then $\lim_{\rho\to\infty} k(\rho, \lambda) = \infty$.

(e) Suppose $F'(w) < cw^r$ for some $r > 0$, $c > 0$, and all large enough w. Then for some $C > 0$,

$$a(\rho, \lambda) \geq C\rho^{1/2r} \tag{19}$$

for sufficiently large ρ.

(f) Let ρ_0^* $(\geq 0) =$ the critical (least) value of ρ beyond which $M(\rho,k) > 0$ for some k (unrestricted by (15)), and for $\rho > \rho_0^*$, let $\bar{k}(\rho) > 0$ be the least positive value of k at which $M(\rho,k)$ is maximized (neither ρ_0^* nor $\bar{k}$ depend on λ). For $\rho > \rho_0^*$,

$$0 < \bar{k}(\rho) = \liminf_{\lambda\to\infty} k(\rho,\lambda) \leq \limsup_{\lambda\to\infty} k(\rho,\lambda) < \infty \tag{20}$$

and

$$u_0 \leq \liminf_{\lambda\to\infty} a(\rho,\lambda) \leq \limsup_{\lambda\to\infty} a(\rho,\lambda) < \infty. \tag{21}$$

Commentary.

1. Items (a) and (b) are threshold results, showing that nontrivial global minimizers exist for $\rho > \rho^*(\lambda)$ $(\geq \rho^*(\lambda)$ in the case $u_0 > 0)$ and not for $\rho < \rho^*(\lambda)$. Moreover, these minimizers are given explicitly. In Lemma 2 below, it will be shown that in typical cases, these are the only global minimizers, although other stable stationary solutions may exist, as shown in Sec. 2.3.

One can prove that these minimizers satisfy a somewhat stronger stability statement in the case $\frac{d^2}{du^2}F(u^2) > 0$ for $u = a$. Namely, if ϕ is a minimizer of this type and ϵ is any positive number, then for some $\delta(\epsilon) > 0$, any solution $u(t)$ with $\|u(0) - \phi\| < \delta$ satisfies $\|u(t) - \epsilon\| < \epsilon$ for all $t > 0$. Here the norm is in L^2. We omit the details.

2. Items (c), (d), and (e) give information about the amplitude and minimal period of the global minimizers which will have been constructed. The amplitude a grows without bound as $\rho\to\infty$, and under a reasonable additional assumption (18) the wavenumber k does as well. Thus the minimal wavelength shrinks. The relation (19) gives a more precise lower bound for the rate of growth of the amplitude. For the case $F(w) = (w - u_0^2)^2$, we have $a \geq O(\rho^{1/2})$ as $\rho\to\infty$. It should be noted that this lower bound depends only on the function F. It was shown in [65] that such a bound may sometimes be improved by taking into consideration the operators A and B as well. In fact in the case of the fourth order problem given in Sec. 3.6.2 below and this same convex function F, those authors established that the amplitude grows at least at the rate $O(\rho)$ as $\rho\to\infty$.

As ρ approaches its threshold value ρ^* from above, the amplitude approaches the position u_0 of F's well. The behavior of the wavelength for ρ near ρ^* can also easily be obtained.

3. Item (f) explores the effect on the global minimizer when the size of the basic period interval becomes very large. We conclude that there is no important effect. The wavenumber k and amplitude a are bounded above and below independently of λ. An examination of the proof will show, in fact, that in typical cases they approach finite limits as $\lambda\to\infty$. This means that the structure of stationary patterns is little affected by the domain size λ, when the latter is large.

4. It will be shown in Lemma 1 below that when $u_0 > 0$ the family of exponential minimizers which exist for $\rho \geq \rho^*$ can be continued as stationary solutions of (12) of exponential type for an interval of values of $\rho < \rho^*$ as well; however the extended solutions are not global minimizers. The question of their stability is considered in Sec. 2.3; see also the comment following Lemma 1 below.

2.2.2 Proofs

Lemma 1 *Let $u_{m,a}(x) = ae^{ik_m x}$, $a > 0$ real, k_m given by (15). There exist functions $a^*(\rho,\lambda)$ and $\rho_1(\lambda) \leq \rho^*(\lambda)$ such that for some m, $u_{m,a}$ is a stationary solution of (12) if $\rho > \rho_1$ and $a = a^*$. Also $a^* > u_0$ if $\rho > \rho^*(\lambda)$, $a^*(\rho^*,\lambda) = u_0$, and $a^*(\infty,\lambda) = \infty$.*

Proof Substituting this function into (12), we obtain the following necessary and sufficient condition for it to be a solution: either $a = 0$ or

$$M(\rho, k_m) - 2F'(a^2) = 0. \tag{22}$$

Let $F_c(w),\ \ w > u_0^2$, be the greatest monotone increasing function with $F_c(w) \leq F(w)$ for every w in that range. Its graph is the convex hull of that of F. We have that $F_c'(w)$ is nondecreasing, and by (11), $F_c'(\infty) = \infty$.

The range of $2F'(w)$ contains $[\gamma, \infty)$ (see (14)), the same as that of $2F_c'(w)$. Thus there exists a solution such that

$$F'(a^2) = F_c'(a^2), \tag{23}$$

provided that $M(\rho, k_m) \geq \gamma$ for some m, which is true if and only if $\rho \geq \rho^*(\lambda)$.

Moreover, the quantity $2F'(a^2)$ takes on all negative values in some interval $[-\alpha, 0]$ as a^2 ranges from 0 to u_0^2. Let $\rho_1(\lambda)$ be such that for each $\rho \in (\rho_1, \rho^*)$, $M(\rho, k_m) \in (-\alpha, 0)$ for some $m(\rho)$. There exist exponential solutions for this range of ρ as well. (Some of them satisfy a linear stability criterion; but generally their minimal wavelength is λ itself, so they are almost constant when λ is large.) This proves the existence part of the lemma. The stated properties of a^* follow from (22).

Lemma 2 *When $\rho \geq \rho^*(\lambda)$, there exists a finite integer $m = m^*(\rho, \lambda)$ which maximizes $M(\rho, k_m)$. It can be chosen so that $\lim_{\rho \downarrow \rho^*} m^*(\rho, \lambda) = m_0 < \infty$. If $\rho \geq \rho^*(\lambda)$ and a satisfies (22) with $m = m^*$ and (23), then $u_{m^*,a}$ is a global minimizer of E.*

Proof It follows from **A2** (10) that for any fixed ρ, $M(\rho, k) < 0$ for k large enough. Therefore when maximizing $M(\rho, k_m)$ over m, we need consider only a finite number of values of m, namely those for which $M > 0$. This establishes the first two statements of the lemma.

We shall show that $u_{m^*,a}$ is a global minimizer of E when $a \geq u_0$. In the following, we drop the subscripts on u, let v be any function in $\mathcal{D}(A) \cap L_\infty$, and calculate $E[u + v]$.

We find that

$$\langle A(u + v), u + v\rangle = \langle Au, u\rangle + 2\Re\langle Au, v\rangle + \langle Av, v\rangle, \tag{24}$$

and the same for B. We have chosen the amplitude a so that (23) holds, which means that

$$F(a^2 + w_1) \geq F(a^2) + w_1 F'(a^2)$$

for all w_1. Choosing $w_1 = 2\Re(u\bar{v}) + |v|^2$, we note that $a^2 + w_1 = |u + v|^2$. Hence

$$\int_0^\lambda F(|u + v|^2)dx \geq \lambda F(a^2) + 2F'(a^2)\int_0^\lambda \Re(u\bar{v})\, dx + F'(a^2)\int_0^\lambda |v|^2 dx. \tag{25}$$

Combining these expressions, we obtain

$$E[u + v] \geq E[u] - 2\Re\langle[Au - \rho Bu - 2F'(a^2)u], v\rangle + \tilde{E}[v], \tag{26}$$

where

$$\tilde{E}[v] = E_0[v] + \frac{1}{\lambda}F'(a^2)\int_0^\lambda |v|^2 dx,$$

$$E_0[v] = -\frac{1}{2}\langle Av, v\rangle + \frac{\rho}{2}\langle Bv, v\rangle. \tag{27}$$

The expression $Au - \rho Bu - 2F'(a^2)u$ in the middle term of (26) vanishes because u is a solution of (12). We conclude that u is a global minimizer of E if $\tilde{E}[v] \geq 0$ for all v.

We may express $\tilde{E}[v] = -\frac{1}{2}\langle Lv, v\rangle$, where L is the linear self-adjoint operator given by

$$Lv = Av - \rho Bv - 2F'(a^2)v.$$

Therefore $\tilde{E}$ will be positive, as desired, if all eigenvalues μ of L are nonpositive. The eigenvalue equation is

$$Av - \rho Bv - 2F'(a^2)v = \mu v. \tag{28}$$

This may be solved by Fourier expansion over the period interval $(0, \lambda)$. It therefore suffices to look for solutions of the form $v = e^{ik_m x}$, k_m given by (15). We obtain the set of eigenvalues

$$\mu_m = M(\rho, k_m) - 2F'(a^2), \quad m = 1, 2, \ldots \tag{29}$$

We subtract (22) with $k = k_{m^*}$ from this to obtain

$$\mu_m = M(\rho, k_m) - M(\rho, k_{m^*}). \tag{30}$$

Therefore since $m = m^*$ was chosen to maximize $M(\rho, k_m)$, we obtain from (30) that all eigenvalues satisfy

$$\mu \leq 0,$$

and hence $E[u+v] \geq E[u]$ for all admissible v. Thus u_{a,m^*} is a global minimizer.

It will be the only one if $E[u+v] > E[u]$ for $v \neq 0$. This can be verified in case m^* is unique and $F(w)$ is strictly convex.

Remark: Values of m which do not maximize $M(\rho, k_m)$ generate stationary solutions which may be stable but not global minimizers. This issue is taken up in Sec. 2.3.

Lemma 3 *If $\rho \leq \rho^*$, then*

$$E[u] \geq 0 \tag{31}$$

for any u in the domain of E. Moreover if equality holds in (31), then necessarily $|u| \equiv u_0$. If equality holds and $\rho < \rho^*$, then $u = const.$

Proof By (9), (27), $E[u] \geq E_0[u]$. The form E_0 is positive if all the eigenvalues of $-A + \rho B$ are nonnegative, i.e. all the eigenvalues of $A - \rho B$ are ≤ 0.

These latter eigenvalues can be found by using exponential functions, and are given by (29) with the last term set equal to zero, i.e. $\mu_m = M(\rho, k_m)$. But when $\rho \leq \rho^*$, this is nonpositive for all m. This completes the proof of the first statement. If equality holds in (31), the last term in (9) vanishes, so that $F(|u|^2) \equiv 0$, hence $|u|^2 = u_0^2$. If moreover $\rho < \rho^*$, then also $\langle Au, u\rangle = 0$, so that $u =$ const. by **A3**.

Proof of Theorem 1

Parts (a) and (c) follow directly from Lemmas 1 and 2, and part (b) from Lemma 3.

Consider now part (d). Recall that our construction in Lemma 2 has $m = m^*$, so that $M(\rho, k_{m^*})$ is maximal. If the assertion in (d) were not true, then there would exist a sequence $\rho_n \to \infty$ and an integer $\bar{m}$ with $m^*(\rho_n, \lambda) = \bar{m}$ for all n. Since m^* maximizes M, we have that for every $m > \bar{m}$ and every n, $M(\rho_n, k_m) \leq M(\rho_n, k_{\bar{m}})$. Thus from (13), setting $\Delta\hat{A} = \hat{A}(k_{\bar{m}}) - \hat{A}(k_m)$, $\Delta\hat{B} = \hat{B}(k_{\bar{m}}) - \hat{B}(k_m)$, we get

$$\rho_n \Delta\hat{B} \leq \Delta\hat{A}.$$

However, by the condition (18), we can always choose a number m so that $\Delta\hat{B} > 0$. Letting $n \to \infty$ gives us then a contradiction, which proves item (d).

Part (e): From (22) and (23) and the hypothesis, we have

$$M(\rho, k_{m^*}) < ca^{2r}.$$

Now let m be any fixed number > 0. We have $M(\rho, k_{m^*}) \geq M(\rho, k_m) \equiv C_1 + C_2\rho$, where the C_i depend on k_m and $C_2 = \frac{1}{2}|\hat{B}(k_m)| > 0$. Thus $a^{2r} > C_1' + C_2'\rho$. This yields (19).

Consider now part (f). Let $I(\rho)$ be the bounded closed interval on the k-axis where $M(\rho, k) \geq 0$.

For all λ, the maximization of $M(\rho, k_m)$ over m is the maximization over a discrete set of values of k in $I(\rho)$, and that in the maximization of $M(\rho, k)$ is taken over all of $I(\rho)$. Therefore the lim sup and lim inf appearing in (20) lie in I, and in fact $\liminf_{\lambda\to\infty} k_{m^*} = \bar{k} > 0$. This proves (20). Finally, (21) follows from this and (22).

This completes the proof of the theorem.

2.3 Linear stability of solutions which are not minimizers

We now return to the linear stability analysis of solutions $u_0 = ae^{ikx}$ of (12). The procedure in [69] applies. It is with regard to solutions on the entire real line, rather than on a finite period interval, as in our context.

Recall (5),(6) for the perturbation ϕ. In our case, the equation for ϕ takes the form

$$\phi_t = A(\phi e^{ikx})e^{-ikx} - \rho B(\phi e^{ikx})e^{-ikx} - 2\phi F'(a^2) - 4a^2 \Re\phi F''(a^2). \qquad (32)$$

Seeking solutions of this equation in the form (6) with scalar C_i, we find the following pair of equations for C_1 and $\bar{C}_2$:

$$\begin{aligned} C_1[\sigma - M(\rho, k+\ell) + 2F' + 2a^2F''] + 4a^2F''\bar{C}_2 = 0, \\ \bar{C}_2[\sigma - M(\rho, k-\ell) + 2F' + 2a^2F''] + 4a^2F''C_1 = 0, \end{aligned} \qquad (33)$$

where the functions F', F'' are evaluated at a^2. The determinant condition for the existence of nontrivial solutions leads to the following second order equation for σ, where we set $M_\pm(k,\ell) = M(\rho, k) - M(\rho, k\pm\ell)$ and $\Sigma(k,\ell) = M_+(k,\ell) + M_-(k,\ell)$:

$$\sigma^2 + \sigma(4a^2F'' + \Sigma) + M_+M_- + 2a^2F''\Sigma = 0. \qquad (34)$$

Here we have used (22).

The larger of the real parts of the two roots of this equation is ≤ 0 for all ℓ iff

$$4a^2F''(a^2) + \Sigma \geq 0 \quad \text{and} \quad -2a^2F''\Sigma \leq M_+M_- \quad \text{for all } \ell. \qquad (35)$$

For stability, (35) must hold. This is our **linear stability criterion**. It is seen that stability cannot occur if $F''(a^2) < 0$, because $\Sigma\to 0$ as $\ell\to 0$, and the first inequality would be violated for small ℓ. Similarly, it can be seen that if $F''(a^2) = 0$, stability can only happen if the given value of k maximizes $M(\rho, k)$.

An important case is when the index k and the function $M(\rho, k)$ are such that $\Sigma > 0$ for all $\ell \neq 0$ (it is always the case that $\Sigma > 0$ for large ℓ). This is true if the concave hull of the graph of the function $M(\rho, k)$ touches the graph itself at the chosen point k and the concavity is strict there. Then from (35) we obtain a sufficient condition for stability, in terms the following function $R(\rho, k)$ (which might be $+\infty$), defined simply in terms of the function M and number k:

$$R(\rho, k) = \sup_{\ell\neq 0} \frac{-(M(\rho,k) - M(\rho,k+\ell))(M(\rho,k) - M(\rho,k-\ell))}{(M(\rho,k) - M(\rho,k+\ell)) + (M(\rho,k) - M(\rho,k-\ell))}. \qquad (36)$$

Then we have the following **linear stability criterion in the case $\Sigma(k,\ell) > 0$ for all $\ell \neq 0$:**

$$R(\rho, k) \leq 2a^2 F''(a^2). \tag{37}$$

When it is not true that $\Sigma > 0$ for all $\ell \neq 0$, then we revert to the above more general criterion (35).

The criterion is easily modified to handle problems in a finite period interval λ, as we were doing. Then the numbers ℓ are restricted to be multiples of $\frac{2\pi}{\lambda}$. All the above continues to hold.

In the case of the global minimizers we have been considering, k has been taken to maximize the function $M(\rho, k)$ under the restriction that k be of the form (15), and we also have $F''(a^2) \geq 0$. Therefore the quotient in (36) is negative and the condition (37) clearly holds, as expected. But (37) also indicates that other solutions, not global minimizers, may be stable as well. In fact when $F'' > 0$, values of k which are near but not at the maximizer for M satisfy the condition.

3 Nonconserved gradient flows for real-valued functions

The foregoing theory falls short in relevance to many applied problems, because it allows complex-valued minimizers; moreover the class of nonlinearities F is restricted to functions of $|u|^2$ alone and the theory is one-dimensional. We may extend the theory to partially remove these difficulties while losing some completeness of our conclusions, by (1) restricting the class of admissible functions competing for the minimization to real-valued functions, (2) allowing more general nonlinearities, and (3) extending the theory to higher space dimensions (although if A and B are isotropic, our criteria for the existence of patterns differ little from those in the 1-D case).

In the N-dimensional context, the operators A and B act on λ-periodic functions $u(x)$, where now $x = (x_1, x_2, \ldots x_N)$ and $\lambda = (\lambda_1, \lambda_2, \ldots \lambda_N)$. The assumptions **A1** to **A3** of Sec. 2.2 are still assumed, with the obvious notational changes: $k = (k_1, k_2, \ldots, k_N)$, e^{ikx} means $e^{ik\cdot x}$, and the limit in (10) is taken as $|k| \to \infty$. Let $\mathcal{D}_r$ be the set of real-valued functions in $\mathcal{D}(A)$.

We restrict u to be real-valued, and consider now energy functionals of the form

$$E[u] = -\frac{1}{2}\langle Au, u\rangle + \frac{\rho}{2}\langle Bu, u\rangle + \frac{1}{|\Lambda|}\int_\Lambda H(u)dx, \tag{38}$$

where the C^1 real function H has a minimum of 0 at some value $u = u_0$ (if it attains this minimum at more than one point, let u_0 be the maximal one). The integral in (38) is the integral over one period cell Λ, and $|\Lambda|$ is the measure of that cell.

We also assume that H grows superquadratically as $|u| \to \infty$:

$$\lim_{|u|\to\infty} \frac{H(u)}{u^2} = \infty. \tag{39}$$

The evolution equation (8) now takes the form

$$u_t = Au - \rho Bu - H'(u), \tag{40}$$

and the minimizers ϕ of E in $\mathcal{D}_r \cap L^\infty$ are stable λ-periodic solutions of

$$A\phi - \rho B\phi - H'(\phi) = 0. \tag{41}$$

In the real valued case we do not know the minimizers explicitly, and this diminishes the completeness of our results. For one thing, the linear stability analysis of Sec. 2.3 is not applicable to patterns which are not sinusoidal, as they will be in this section; we therefore restrict attention completely to global minimizers of the energy.

A second point is that the existence of minimizers is not always clear. We shall assume that for each ρ, λ there is $\phi^\rho \in \mathcal{D}_r(A) \cap L_\infty$ such that

$$\min_{u\in\mathcal{D}_r(A)\cap L_\infty} E[u] = E[\phi^\rho] \tag{42}$$

Remark

In general condition (42) might be difficult to verify without making additional assumptions on the operators A, B. On the other hand in special cases proving (42) usually involves checking that E is coercive and weakly lower semicontinuous. The condition (39) will be relevant to establishing coercivity. We illustrate this by considering case (b) near the beginning of Sec. 2.2:

$$Au = u_{xx}, \quad Bu = G * u - u, \quad H(u) = F(u^2),$$

resulting in the equation (85) below. Here $G * u$ represents convolution of u with a nonnegative kernel with unit integral. The corresponding energy is

$$E[u] = \frac{1}{\lambda}\int_0^\lambda \frac{1}{2}|u'|^2 dx - \frac{1}{\lambda}\frac{\rho}{4}\int_{-\infty}^{\infty}\int_0^\lambda G(y)|u(x+y)-u(x)|^2 dx dy + \frac{1}{\lambda}\int_0^\lambda F(|u|^2)dx. \tag{43}$$

We set χ_D to be the characteristic function of the set D. For each $K > 0$ there exist constants $C_1(K,\rho)$, $C_2(\rho)$ such that

$$\frac{\rho}{4}\int_{-\infty}^{\infty}\int_0^\lambda G(y)|u(x+y)-u(y)|^2\, dx dy \le C_1(\rho, K) + C_2(\rho)\int_0^\lambda u^2(x)\chi_{\{u^2>K\}}\, dx. \tag{44}$$

On the other hand

$$\int_0^\lambda F(u^2) \ge \int_0^\lambda F(u^2)\chi_{\{u^2>K\}}. \tag{45}$$

Using the fact that $F'(w) \to \infty$ as $w \to \infty$ we conclude that for sufficiently large K we have $[F(u^2) - u^2]\chi_{\{u^2>K\}} > 0$ and thus combining (44), (45) we see that there exists a constant $C_3(K,\rho)$ such that

$$E[u] \ge \frac{1}{\lambda}\left[\int_0^\lambda \frac{1}{2}|u'|^2 - C_3(K,\rho)\right] \tag{46}$$

and thus E is coercive in H^1.

From the embedding $C^\alpha(0,\lambda) \hookrightarrow H^1(0,\lambda)$, $\alpha \in [0,1)$ we conclude that E is weakly lower semicontinuous with respect to $H^1_{per}(0,\lambda)$ norm, where $H^1_{per}(0,\lambda)$ denotes the space of λ-periodic H^1 functions.

From (46) and the weak lower semicontinuity we can verify (42) by a fairly standard argument. We observe that the minimizers are in fact smooth.

3.1 Dependence of minimizers on ρ

Associated with the function H, we define two other functions H^* and H_0.

In accordance with our periodicity constraint, we consider (as in (15)) wavenumber vectors

$$k^m = (k_1^{m_1}, k_2^{m_2}, \ldots k_N^{m_N}) \tag{47}$$

with

$$k_i^{m_i} = \frac{2\pi m_i}{\lambda_i}, \tag{48}$$

not all of the m_i vanishing. Let $|m|$ be the number of integers $i \leq N$ with $m_i \neq 0$.

When k is of this form, it is clear that the integral

$$\frac{1}{|\Lambda|} \int_\Lambda H(a \prod_i \cos(k_i^{m_i} x_i) + b) dx = H^*(a, b, |m|) \tag{49}$$

depends only on a, b, and $|m|$, i.e. how many indexes m_i vanish. This is because when $m_i \neq 0$, the integrand is nontrivially periodic in x_i of period $\frac{\lambda_i}{m_i}$. For example if all the m_i except one vanish ($|m| = 1$), we have that the integral on the left of (49) is equal to $\frac{1}{\lambda_1} \int_0^{\lambda_1} H\left(a \cos\left(\frac{2\pi m_1}{\lambda_1} x_1\right) + b\right) dx_1 = \int_0^1 H(a \cos(2\pi x) + b)\, dx$, and if $|m| = 2$, it is

$$\int_0^1 \int_0^1 H(a \cos(2\pi x_1) \cos(2\pi x_2) + b)\, dx_1 dx_2.$$

In any case, the function H^* is even in a and has a minimum of 0, attained at $(a, b) = (0, u_0)$.

We now set

$$H_0(a, |m|) = \min_b H^*(a, b, |m|), \tag{50}$$

with the minimum attained at a value $b = b^*(a, |m|)$.

Let

$$M^*(|m|) = \inf_{a>0} \frac{4H_0(a, |m|)}{a^2}, \tag{51}$$

which is either attained at a positive value a^* of a (finite because of the superquadratic growth of H) or approached as $a \to 0$ (in which case we set $a^* = 0$).

It is important to emphasize that M^* depends only on the nonlinearity H and $|m|$. For example, in the case $H(u) = (1 - u^2)^2$ and $N = 1$, it turns out [43] that $M^* = .899$. Essentially this same calculation, leading to the same sufficient condition for the existence of stable patterns, was done in the context of the equation (75) below by Mizel, Peletier, and Troy [65].

Recalling the definition (47) of k^m, we define

$$\bar{\rho}(\lambda) = \inf\{\rho : M(\rho, k^m) > M^*(|m|) \text{ for some } m\} \tag{52}$$

and

$$\rho_0^*(\lambda) = \inf\{\rho : M(\rho, k^m) > 0 \text{ for some } m\}. \tag{53}$$

Note that if $H(u) = F(u^2)$ and $u_0 > 0$, the number ρ_0^* coincides with ρ^* given by (14).

Theorem 2 *There exists a number $\rho_c(\lambda) \in [\rho_0^*(\lambda), \bar{\rho}(\lambda)]$ such that*

(a) for each $\rho < \rho_c$, there exists no nontrivial global minimizer of the functional E (38) in the class of real-valued functions;

(b) for each $\rho > \rho_c$, there exists such a nontrivial real global minimizer ϕ^ρ of E in $\mathcal{D}_r \cap L^\infty$ with

$$E[\phi^\rho] < 0. \tag{54}$$

Let $E_1(\rho) = -\frac{1}{4}(a^*)^2[M(\rho, k^m) - M^*(|m|)]$, where m is chosen and fixed so that $M(\rho, k^m) > 0$ for some $\rho = \rho_0$. Since M is an increasing function of ρ, this will be true for all $\rho > \rho_0$ as well. For $\sigma > 0$, let

$$P(\sigma) = \max_{|u| \le \sigma}(uH'(u) - 2H(u)).$$

It can be shown that (39) implies $\lim_{\sigma\to\infty} P(\sigma) = \infty$; in fact if $uH'(u) - 2H(u)$ were bounded for large u by some number K, then integrating the inequality $uH'(u) - 2H(u) < K$ would imply that $\frac{H(u)}{u^2}$ is bounded.

Theorem 3 *For $\rho > \rho_c$, let ϕ^ρ be a minimizer of E, and $a(\rho) = \max|\phi^\rho(x)|$. Then*

$$P(a(\rho)) \ge -2E_1(\rho). \tag{55}$$

Since $-E_1$ increases linearly with ρ and $P(a)$ increases to ∞ as $a\to\infty$, we see that (55) provides a lower bound on the amplitude which grows toward ∞ as $\rho\to\infty$.

Corollary 1 *Assume $H(u) = c|u|^r + O(|u|^{r-1})$ as $u\to\infty$, $r > 2$, and that the corresponding differentiated relation $H'(u) = rcu|u|^{r-2} + O(|u|^{r-2})$ holds. Then*

$$a(\rho) \ge C\rho^{1/r} \tag{56}$$

for large ρ, where C depends only on the function H.

Corollary 2 *In the case $H(u) = (1-u^2)^2$, we have, for $\rho > \rho_c$,*

$$\max_x |\phi^\rho(x)| \ge 1. \tag{57}$$

Proof of Theorem 2

Any global minimizer ϕ^ρ with $E[\phi^\rho] < 0$ must be nontrivial, because constants have $E \ge 0$. To emphasize dependence on ρ, we write $E[u] = E_\rho[u]$. Let

$$\rho_c = \inf\{\rho : E_\rho \text{ has a minimizer } \phi^\rho \text{ with } E_\rho[\phi^\rho] < 0\}.$$

If $\rho_c < \infty$, the assertion (b) holds by virtue of this definition and the fact that $E_\rho[\phi]$ is a decreasing function of ρ.

Now suppose that for some number $\rho_0 < \rho_c$, E_{ρ_0} has a nontrivial global minimizer ϕ^{ρ_0}. Then $E_{\rho_0}[\phi^{\rho_0}] = 0$, since the minimum of E_ρ for every ρ is always nonpositive. Since ϕ^{ρ_0} is nontrivial, we have from Assumption **A3** that $\langle B\phi^{\rho_0}, \phi^{\rho_0}\rangle < 0$. Hence $E_\rho[\phi^{\rho_0}]$ is strictly decreasing in ρ, so that $E_\rho[\phi^{\rho_0}] < 0$ for $\rho_0 < \rho < \rho_c$. The global minimizers for these values of ρ must be nontrivial, contradicting the definition of ρ_c. Thus part (a) follows.

We show that ρ_c lies in the indicated interval. If $\rho < \rho_0^*$, $M(\rho, k^m) < 0$ for all m, so that the operator $A - \rho B$ is negative definite, and since $H(u) > 0$, it follows from (38) that $E[u] \ge 0$ for all u, hence by our definition of ρ_c, $\rho \le \rho_c$, and we conclude that $\rho_c \ge \rho_0^*$.

To show that $\rho_c \le \bar\rho$, we choose $a = a^*$ (51) and $b = b^*(a^*)$ (50) to obtain from (38), (51) that

$$E[a^* \prod_i \cos(k_i^{m_i} x) + b^*] = -\frac{1}{4}(a^*)^2[M(\rho, k^m) - M^*(|m|)] < 0 \tag{58}$$

for some m for $\rho > \bar\rho$. Hence the minimizer ϕ^ρ for such ρ has negative energy and must be nontrivial, so that $\rho \ge \rho_c$.

Proof of Theorem 3:

Since ϕ^ρ is a minimizer, it satisfies (41). Take the scalar product of (41) with ϕ^ρ:

$$\langle (A-\rho B)\phi^\rho, \phi^\rho\rangle - \frac{1}{|\Lambda|}\int_\Lambda \phi^\rho H'(\phi^\rho)dx = 0, \tag{59}$$

so that

$$2E[\phi^\rho] + \frac{1}{|\Lambda|}\int_\Lambda \left(\phi^\rho H'(\phi^\rho) - 2H(\phi^\rho)\right) dx = 0. \tag{60}$$

Note from (58) that $E_1(\rho) = E[a^* \prod_i \cos(k_i^{m_i}x) + b^*]$. Since ϕ^ρ is a minimizer, we have $E[\phi^\rho] \le E_1(\rho)$, hence from (60),

$$2E_1(\rho) + \frac{1}{|\Lambda|}\int_\Lambda \left(\phi^\rho H'(\phi^\rho) - 2H(\phi^\rho)\right) dx \ge 0. \tag{61}$$

Since $P(a(\rho)) \ge \frac{1}{|\Lambda|}\int_\Lambda \left(\phi^\rho H'(\phi^\rho) - 2H(\phi^\rho)\right) dx$, we obtain (55).

This completes the proof.

Proof of corollaries

In the case of Cor. 1, we have

$$P(\sigma) = \max_{|u|\le\sigma}\left(rc|u|^r - 2c|u|^r + O(|u|^{r-1})\right) = (r-2)c\sigma^r + O(\sigma^{r-1}) \ \ (\sigma\to\infty).$$

We also have $E_1(\rho) \ge c_1\rho - c_2$ for some positive constants c_j. Thus (56) follows easily. We omit the proof of Cor. 2.

3.2 Bifurcating solutions

Continuing with the energy functional given by (38), we assume the dimension is one. Suppose $u_0 = 1$. Again, we consider real-valued steady state solutions $\phi(x)$ of (40), periodic in x with period λ. We set $f(u) = H'(u)$. There is of course the trivial solution $\phi(x) \equiv 1$, which is stable for $\rho = 0$. As ρ increases, however, there will be a point where it loses its stability. In fact, an easy linear stability analysis shows that it becomes unstable as soon as $M(\rho, k_m)$, for some m, surpasses $f'(1)$ (k_m is given by 15). Then generally other solutions will arise, some of them being stable. In this section we establish the existence of an infinite number of bifurcating solution branches, ρ being the bifurcation parameter.

For the purpose of this section we shall make the following additional assumption on the operator A:

A4.

$$-\sum_{m=1}^{\infty}\frac{1}{\hat{A}(k_m)} < \infty$$

It is easy to check that **A4** holds for the operator $A = \Delta$.

Theorem 4 *Let* $\rho_m(\lambda) = \inf\{\rho :\ M(\rho, k_m) > f'(1)\}$.

For each positive integer m, ρ_m *is a bifurcation point of real-valued steady states from the trivial solution* $\phi \equiv 1$.

Proof of Theorem 4

Write $\phi = 1 + \psi$. Let $Z = L_2 \cap$ span $\{\cos(k_j x,\ j \geq 0\}$. Note that $\psi \in Z$ implies $\psi(x) = \psi(\lambda - x)$. Furthermore we define $X = \mathcal{D}_r(A) \cap Z \cap C^0(0,\lambda)$. Now X equipped with the norm $\|\psi\|_X^2 = \|A\psi\|^2 + \|B\psi\|^2 + \|\psi\|^2 + \|\psi\|_{C^0}^2$ is a Banach space.

Set

$$\mathcal{F}(\psi, \rho) = (T - \rho B)\psi - h(\psi) \tag{62}$$

where $T\psi = A\psi - f'(1)\psi$, $h(\psi) = f(1+\psi) - f'(1)\psi$. If $\psi \in X$ has the Fourier expansion

$$\psi = \sum a_j \cos(k_j x) \tag{63}$$

then

$$A\psi = \sum a_j \hat{A}(k_j) \cos(k_j x) \in Z;$$

and a similar formula holds for B. For $\psi \in X$ we also have that $h(\psi) \in C^0(0,\lambda)$ hence $h(\psi) \in Z$. Using this and the smoothness of f it follows that $\mathcal{F} \in C^2(X; Z)$. Moreover T and B are bounded linear operators from X into Z. It can be checked easily that $h(0) = Dh(0) = 0$ (using the Frechet derivative D). We shall use the well known theorem of Crandall and Rabinowitz [28] to show the existence of solutions of

$$\mathcal{F}(\psi, \rho) = 0 \tag{64}$$

bifurcating from simple eigenvalues of the pair (T, B) at $\psi = 0,\ \rho = \rho_m$.

Recall that $\rho \in \mathbf{R}$ is a simple eigenvalue of the pair (T, B) if

$$\dim \mathcal{N}(T - \rho B) = 1 = \operatorname{codim} \mathcal{R}(T - \rho B); \tag{65}$$

$$[B\mathcal{N}(T - \rho B)] \oplus \mathcal{R}(T - \rho B) = Z, \tag{66}$$

where $\mathcal{N}, \mathcal{R}$ denote the null space and the range of $T - \rho B$ respectively [25].

Let $\psi \in \mathcal{N}(T - \rho B)$ with Fourier expansion as in (63). We then have

$$\sum a_j [\hat{A}(k_j) - f'(1) - \rho \hat{B}(k_j)] \cos(k_j x) = 0$$

hence

$$\sum a_j [\hat{A}(k_j) - f'(1) - \rho \hat{B}(k_j)] \langle \cos(k_j x), \cos(k_m x) \rangle = 0$$

It follows that $\psi \in \mathcal{N}(T - \rho B)$ if and only if, for some m, $\psi \in$ span $\{\cos(k_m x)\}$ and $\rho = \rho_m$. Thus $\dim \mathcal{N}(T - \rho B) = 1$.

We fix $n \geq 1$ and $\rho = \rho_n$. We will show that $\operatorname{codim} \mathcal{R}(T - \rho B) = 1$. Let

$$\psi = \sum_{m \neq n} b_m \cos(k_m x).$$

Since A is closed, a function $\eta \in Z$ defined by

$$\eta = \sum_{m \neq n} a_m \cos(k_m x), \qquad a_m = \frac{b_m}{\hat{A}(k_m) - f'(1) - \rho \hat{B}(k_m)}$$

solves the equation $(T - \rho B)\eta = \psi$. From **A2** we have for all sufficiently large m

$$|a_m| \leq 2 \frac{|b_m|}{-\hat{A}(k_m)} \leq 2 \frac{\|\psi\|}{-\hat{A}(k_m)},$$

hence using **A4** we conclude that $\eta \in \mathcal{D}(A) \cap C^0(0,\lambda)$. Thus we have $\eta \in X$ and span $\{\cos(k_m x), m \neq n\} \in \mathcal{R}(T - \rho B)$. On the other hand if $\psi \in \mathcal{R}(T - \rho B)$ then for some $\eta \in X$ we have $\psi = (T - \rho B)\eta \in$ span $\{\cos(k_m x), m \neq n\}$. It follows that span $\{\cos(k_m x), m \neq n\} = \mathcal{R}(T - \rho B)$.

This establishes (65). Since $B[\cos(k_m x)] = \hat{B}(k_m)\cos(k_m x)$, (66) holds as well.

The assertion of the theorem then follows by [28, 29].

3.3 Priority of small vs. large patterns

We now have two criteria for the appearance of patterned solutions, when the nonlinearity is bistable with stable zero at (say) $u_0 = 1$: the one arising from Thm. 4, namely $M(\rho, k_m) > f'(1)$, and that arising from Thm. 2, namely $M(\rho, k_m) > M^*$ (in the case $N = 1$). The second is only a sufficient condition, and the first provides bifurcating solutions whose stability would have to be checked.

As the control parameter ρ increases in magnitude, it will be interesting to determine which of these two criteria is first satisfied. This depends simply on the relative magnitudes of M^* and $f'(1)$, which in turn depend only on H. Thus this priority will be independent of A or B.

In the case $H(u) = (1 - u^2)^2$, it was indicated following (51) that $M^* = .899\ldots$, whereas $f'(1) = 8$. Therefore in this case, patterns with amplitude around 1 appear much earlier than the solutions bifurcating from 1. The latter will in many cases probably be local but not global minimizers of the energy. It is also interesting to note that the criterion involving $f'(1)$ can be obtained formally by constructing a number M_1^* analogous to (51), but by holding $b = 1$ throughout.

3.4 Dependence of the solutions on λ

We now fix $\rho > \rho_c$ and ask how the minimizers depend on λ. Again for simplicity, we restrict to the dimension $N = 1$, although analogous results hold in general. We suppress the dependence on ρ and write the minimizer ϕ as ϕ_λ. We now emphasize the λ-dependence of the energy functional E by writing it as E^λ. Finally, we write $E(\lambda) = E^\lambda[\phi_\lambda]$.

Since the estimate (55) is independent of λ, we know that the amplitudes of our minimizing patterns remain bounded away from 0 as $\lambda \to \infty$. The following result on the minimal energy holds.

Theorem 5 *For every positive number λ_0, the sequence $E(\nu\lambda_0)$ is nonincreasing in the integer ν and approaches a finite negative limit as $\nu \to \infty$.*

Proof. Any λ_0-periodic function u is also a $\nu\lambda_0$-periodic one; moreover, its $\nu\lambda_0$-energy is identical to its λ_0-energy. This is because (a) $\frac{1}{\lambda}\int_0^\lambda H(u)\,dx$ is clearly the same, and (b) if we expand u in Fourier series $u = \sum_{-\infty}^{\infty} u_n e^{i2n\pi x/\lambda_0}$,
$u_{-n} = u_n$, we see, defining $E_0^\lambda[u] = -\frac{1}{2}\langle(A - \rho B)u, u\rangle$, that

$$E_0^{\lambda_0}[u] = -\frac{1}{2}\sum_{n=-\infty}^{\infty} |u_n|^2 M\left(\rho, \frac{2n\pi}{\lambda_0}\right) = -\frac{1}{2}\sum_{m=-\infty}^{\infty} |\tilde{u}_m|^2 M\left(\rho, \frac{2m\pi}{\nu\lambda_0}\right). \tag{67}$$

where $\tilde{u}_m = u_n$ for $m = \nu n$; $\tilde{u}_m = 0$ otherwise. The right side of (67) is $E_0^{\nu\lambda_0}[u]$. It follows that

$$E(\nu\lambda_0) \le E^{\nu\lambda_0}[\phi_{\lambda_0}] = E^{\lambda_0}[\phi_{\lambda_0}] = E(\lambda_0),$$

which proves the monotonicity of $E(\nu\lambda_0)$.

To show that $E(\nu\lambda_0)$ approaches a limit, it suffices to show that $E^\lambda[u]$ is bounded below, independently of λ and u, for fixed ρ. Let $F(u^2)$ be a nonnegative smooth convex function of u^2 satisfying $F(u^2) \le H(u)$ and (11). Let $\hat{E}$ be the associated energy (9). Thus $\hat{E}[u] \le E[u]$ for all u. In Theorem 1 we found the minimizers of $\hat{E}$, among complex-valued periodic functions, explicitly. They are exponential functions. Their energies are verified to be bounded independently of λ. The minimal energies among real-valued functions are no less than they are among complex-valued ones. Therefore the energies $E[u]$ are likewise bounded below. This completes the proof.

Consider now the minimizer ϕ_{λ_0} for any period interval λ_0. Since it is also a solution of (41) with period $\nu\lambda_0$ for any positive integer N, it is a stationary point for $E^{\nu\lambda_0}$ on that larger interval, but is no longer necessarily a minimizer. If λ_0 is large, however, ϕ_{λ_0} is at worst only weakly unstable with respect to the interval $\nu\lambda_0$, in the sense that solutions of the evolution problem starting near ϕ_{λ_0} move away from ϕ_{λ_0}, if at all, only slowly. Specifically, we have the following result about the L^2 norm of the velocity u_t:

Theorem 6 *Consider the evolution (40), where u is required to be a $\nu\lambda_0$-periodic function of x for each time t, and to satisfy the initial condition $u(x,0) = u_0(x)$. Let $\delta(\lambda)$ be any function of λ approaching 0 as $\lambda\to\infty$. There exists a function $\delta_1(\lambda)$ independent of ν with* $\lim_{\lambda\to\infty}\delta_1(\lambda) = 0$ *such that if $E^{\nu\lambda_0}[u_0] \le E^{\nu\lambda_0}[\phi_{\lambda_0}] + \delta(\lambda_0) = E(\lambda_0) + \delta(\lambda_0)$, then*

$$\int_0^\infty \|u_t(\cdot,t)\|^2 dt \le \delta_1(\lambda_0). \tag{68}$$

Proof. We calculate

$$\frac{d}{dt}E[u(\cdot,t)] = -\langle Au - \rho Bu - H'(u), u_t\rangle = -\|u_t(\cdot,t)\|^2, \tag{69}$$

hence

$$\int_0^\infty \|u_t(\cdot,t)\|^2 dt = E^{\nu\lambda_0}[u_0] - \lim_{t\to\infty} E^{\nu\lambda_0}[u(\cdot,t)] \le E(\lambda_0) + \delta(\lambda_0) - E(\nu\lambda_0).$$

The conclusion (68) follows by the previous theorem 5 and our assumption on δ.

3.5 Further questions

The most important continuing questions about the minima of E (38) have to do with their dependence on λ.

It may be expected that in many cases the global minimizers, or translates of them, will approach some periodic function as $\lambda\to\infty$ uniformly on bounded intervals. If that is true, then the pattern's properties will be more or less insensitive to the size of the domain in x-space.

Simple as this concept may be, apparently the only case which has been succesfully studied from this point of view is that of the functional (74) below and its generalizations to some other second order functionals. The results of Leizarowitz, Marcus, Mizel, and Zaslavski will be reviewed below in Section 3.6.3.

Our theorem 3 establishes a lower bound on the amplitude of ϕ^ρ which is independent of λ; but it is quite possible that no sequence of them, as $\lambda\to\infty$, will approach a periodic stationary solution.

3.6 The Swift–Hohenberg and related equations

3.6.1 Motivation from fluid convection

The Swift-Hohenberg equation, which in its simplest form is

$$u_t = -(\nabla^2 + 1)^2 u + \alpha u - u^3, \tag{70}$$

together with its generalizations, have been, and continue to be, extremely popular as models for various kinds of patterns in nature. Many references can be found in [31] and [27]. Especially interesting are its more recent uses in modeling localized patterns, such as "oscillons" seen in vibrating granular materials [77, 78, 30] (see [76] for a review of various mechanisms capable of producing localized patterns). This equation is sometimes generalized to have complex coefficients;

then traveling wave solutions may exist. The equation is no longer gradient in that case. For examples pertinent to lazers, see [52, 53], and to patterns of swimming microorganisms, [63].

In their original paper [81], Swift and Hohenberg studied Bénard convection with random thermal fluctuations by means of a weakly nonlinear analysis of the Boussinesq equations. Because of the inherent symmetry of the model, they predicted a first-order transition from disordered to ordered state past a critical value of the Rayleigh number, although they comment that this first order transition will be unobservable in their setup because of the smallness of the fluctuations.

The weakly nonlinear analysis leads to a fourth order parabolic differential equation with an added cubically nonlinear integral term with complicated kernel. Making the approximation that the kernel is constant reduces the integral term to an ordinary cubic nonlinearity. The evolution equation which they obtained is of the form

$$\frac{\partial w}{\partial t} = -\alpha \frac{\delta F}{\delta w} + \eta, \tag{71}$$

with free energy given by

$$F[w] = \int d\mathbf{x} \left[w(\mathbf{x}) \left(R + \beta(\nabla^2 + q^2)^2 \right) w(\mathbf{x}) + \gamma w^4 \right]. \tag{72}$$

Here the solution w represents the amplitude of the most unstable mode (a linear combination of fluid velocity and temperature), q is a critical wavenumber, and R, α, β are physical parameters. In fact, R is a scaled Rayleigh number. The last term η is a random forcing term due to thermal fluctuations.

The gradient flow is

$$\begin{aligned} u_t &= -\left(\nabla^4 + 2q^2\nabla^2 + q^4\right) w + \tfrac{R}{\beta} w - \tfrac{1}{4\beta}\gamma w^3 + \eta \\ &= -\nabla^4 w - 2q^2\nabla^2 w - \mu w - K w^3 + \eta, \end{aligned} \tag{73}$$

where $K = \frac{\gamma}{4\beta}$ and

$$\mu = \frac{q^2 - R/\beta}{\delta/4\beta}.$$

In typical cases, $\mu > 0$ so that the function analogous to our $f(u)$ in (1) is monostable (at the origin) rather than bistable. The equation (70) is a scaled version of this.

Returning to (70), we see that it is form (40) with $A = -\nabla^4$, $B = \nabla^2$, $\rho = 1$, and $H(u)$ a quartic nonlinearity. The control parameter is α rather than ρ.

Our gradient flow does not have the fluctuation term η (73).

In [81], attention is drawn to a previous model of Brazovskii [13] which has many of the same characteristics. Brazovskii's model is less grounded in specific differential equations; it is a general model for the transition between a disordered (as for a liquid) to an ordered (as in certain crystals) phase of a material in the presence of fluctuations. Again, a first order transition is found, occurring at a critical value of the relevant parameter which is determined by the criterion that the energy difference between the two phases changes sign.

A comparison of the two energies is in effect the main tool which we have used in our analysis of the appearance of stable patterns.

3.6.2 Motivation from materials science

A fourth order equation similar to (70) was also proposed as a model for nonuniformities in a nonlinearly elastic rod by Coleman, as described in [54]. Consider such a rod of unstressed length

λ, and subject it to tension, sufficient to make its stretched length equal to ℓ. Let $X(x)$, $x \in [0, \lambda]$, be the deformation, so that $X(0) = 0$, $X(\lambda) = \ell$. Let $u(x) = X'(x)$ be the strain. It is postulated that the equilibrium value of $u(x)$ minimizes the energy functional

$$E[u; \lambda] = \int_0^\lambda (\frac{1}{2}u_{xx}^2 - \frac{1}{2}\rho u_x^2 + H(u))\, dx \tag{74}$$

under the length constraint $\int_0^\lambda u(x)\, dx = \ell$. (Here I have translated their notation into that of the present paper.) It is expected that when $\rho > \rho_c > 0$ for some critical value ρ_c, the indefinite nature of the energy will result in the minimizing strain being oscillatory.

3.6.3 Previous and current results

In the paper [54], a detailed mathematical theory is given for the generalization of the problem of minimizing (74) to one obtained when $\lambda \to \infty$, so that the rod occupies the half-line $[0, \infty)$ (analogous results hold for $(-\infty, \infty)$). Moreover, the integrand in (74) is generalized to other functions, still involving first and second order derivatives. An extensive mathematical treatment of the analogous problem with a mass constraint was given in [26, 58, 59, 60, 61]. Such variational problems on an infinite interval are sometimes called "infinite horizon variational problems". Marcus and Zaslavski ([87] and later papers) have further developed the theory of second order infinite horizon problems.

Considerable discussion is given in [54] to the question of how to formulate a reasonable variational problem on the infinite interval, so that the energy is finite for bounded (not necessarily periodic) functions u. Moreover, the minimizer should automatically minimize the energy "locally" in some sense. Finally, the requirement of fixed length clearly is not appropriate; the authors instead impose a given tension.

In [54], the existence of minimizers in their sense was proved, and also the existence of periodic minimizers. The method of proof relies on a local minimization problem making sense: the local problem is to minimize the integral analogous to (74) under the constraint that boundary values for u and u_x are prescribed at the two ends of the interval.

This extension to an infinite interval is designed to exhibit a possibly patterned stable configuration for the strain which is independent of any boundary effects coming from the ends of the rod. In the generality of our problem (40), the approach in [54] is not likely to be possible, because it is not appropriate or possible to prescribe boundary conditions at the ends of a finite interval.

As I mentioned before, an analysis of the corresponding problem on an infinite domain with mass constraint was initiated by Coleman, Marcus, and Mizel [26] and continued by Marcus and Zaslavskii [58, 59, 60, 61]. For both the unconstrained and constrained problem, it was proved in [59] that any sequence of minimizers for the corresponding finite domain problem, in which the length of the domain tends to infinity, has a subsequence converging to a minimizer of the infinite domain problem. Rates of convergence are found and much more.

Many of the results concerning the constrained problem were extended by Marcus and Zaslavski for a much more general family of second order integrands, in [61].

In [60], the authors deal with unconstrained problems and discuss the shape of minimizers on $(0, \infty)$ and in particular the shape of periodic minimizers. All this is done for a wide class of integrands, not just the basic model. In addition they show that generically, in a very precise way, there is uniqueness (up to translation) of the periodic minimizer. This means that by arbitrarily small perturbations of a given integrand one obtains integrands with this uniqueness property.

In a paper under preparation, these same authors obtain the precise behavior of optimal solutions for the unconstrained problem on the half line and show that at infinity each of them converges to a periodic solution.

The existence of periodic stationary solutions of the Swift-Hohenberg equation (70) was given in [27] for small positive values of $\alpha - 1$.

Motivated by [54], extensive further investigations into solutions of

$$u_t = -u_{xxxx} - \rho u_{xx} + u - u^3 \tag{75}$$

were made by Mizel, Peletier and Troy in [65]. The same concept of variational problem for functions on the whole line was used as in [54]. A specific estimate for a critical value $\rho = \rho_1$ was given, beyond which nontrivial minimizers exist. It coincides with the number $\bar{\rho}$ (52) calculated for this example.

Symmetry properties of the periodic minimizers were proved; for example they must be even with respect to maxima. Upper and lower bounds for the minimal energy $E(\rho)$ were given in terms of ρ; in particular it was found that $E(\rho)$ approaches $-\infty$ like the 4th power of ρ. Finally, the authors constructed global branches of stationary periodic solutions of (75), the parameter again being ρ.

3.7 Activator–inhibitor patterns in biology

Activator-inhibitor models are usually pairs of nonlinear reaction-diffusion equations whose nonlinearities have certain monotonicity properties, suggesting that one of the two functions in the solution pair—the activator u— serves to increase the rate of both reactions, and the other—the inhibitor v—serves to inhibit them. With assumptions on the relative magnitudes of the two diffusivities, these systems are often used to model patterns (not necessarily stationary ones) in biology. See, for example, [47, 62] among many references.

A rough intuitive description of the basis of these models is this. Consider a solution of the evolution system which is constant in both space and time. If this uniform distribution is perturbed near a single location by the introduction, say, of an extra amount of u, this additional activator will cause both u and v to be produced at that location in greater amounts. The inhibitor v, however, diffuses away rapidly, spreading its effect over a larger territory. Its inhibiting effect is thereby diminished at the original point of production. As a result, the initial surplus of u can continue to increase locally to form a spike-like inhomogeneity, or pattern. Nonlinear effects prevent it from becoming too large, and a stable spatial pattern results.

In short, unequal diffusivities may cause the uniform distribution to be unstable, with typical instabilities growing to form spatially patterned states.

The competition between the activator and inhibitor influences is not the same as the competition between the stabilizing influence of the operator A and the destabilizing influence of $-\rho B$ that we examined previously (12,40). However, we shall draw a connection now between the two settings.

Consider the system

$$\begin{aligned} u_t &= D_u \nabla^2 u + f(u,v), \\ v_t &= D_v \nabla^2 v + g(u,v), \end{aligned} \tag{76}$$

in which the D's are constants and the derivatives of the reaction terms f and g satisfy, for some $M > 0$,

$$\begin{aligned} &f_u > 0, \quad g_u > 0 \quad \text{for } |u| < M \text{ and all } v, \\ &f_v < 0, \quad g_v < 0 \quad \text{for all } u, v. \end{aligned} \tag{77}$$

Thus an increase in the amount of u serves to enhance both reactions when u is not too large, namely $|u| < M$. Commonly the behavior of at least the function f is different for larger values of $|u|$, it being such as to effectively limit the growth of u. This results in a limitation on the amplitudes of the patterns.

Apparently the first stability analysis of large amplitude patterns for systems in this class was given in [39]. It was for functions f and g of a special type.

The following system of FitzHugh-Nagumo type is an example of an activator-inhibitor system (space, time, and the variable v have been scaled to produce a system with few parameters):

$$\begin{aligned} u_t &= D\nabla^2 u - f(u) - \rho(v-u), \\ \epsilon v_t &= \nabla^2 v + u - v. \end{aligned} \tag{78}$$

Here the function $f(u)$ is decreasing in a finite interval $u \in I$, and an increasing function outside of I. In fact, $\lim_{|u|\to\infty} \frac{f'(u)}{|u|} = \infty$. If D is an $O(1)$ quantity, the parameter $\frac{1}{\epsilon}$ is a measure of the relative magnitude of the diffusivity of v, as well as of the strength of the v-reaction. The parameter ρ represents a measure of the strength of v's inhibition and u's activation of the first reaction.

3.7.1 Fast kinetics

Consider, for simplicity, bounded solutions of (78) in the entire space $\mathbf{R}^3$, in the limit $\epsilon \to 0$. The second equation can be solved for v in terms of u:

$$v = (-\nabla^2 + 1)^{-1} u = G * u, \tag{79}$$

where G is Green's function for the operator $(-\nabla^2 + 1)$. With this, (78) becomes

$$u_t = D\nabla^2 u - \rho(G * u - u) - f(u), \tag{80}$$

which has the form of (40) with

$$Au = D\nabla^2 u, \quad Bu = G * u - u \tag{81}$$

and $H(u) = \int_0^u f(s)\,ds$. The function H has the properties required in Sec. 3, namely superquadratic growth in $|u|$. If $f(u) = 2uF'(u^2)$ for a function $F(w)$ with superlinear growth, it fits into the framework of Sec. 2.2.

Models of this sort were presented in [72, 80, 48]. Models bearing some similarity were studied in [71, 55, 12].

We consider one space dimension, as entirely analogous results hold for higher ones. In 1D, the convolution involves an integral over the whole line with the Green's function kernel $G(y) = \frac{1}{2}e^{-|y|}$, and the function u is periodic with period λ. It is easily checked that $G * u$ is then also periodic with the same period. The operator B in fact satisfies all the requirements of our theory with $\hat{G}(k)$ given below. We have

$$\hat{A}(k) = -Dk^2, \quad \hat{B}(k) = \hat{G}(k) - 1 = \frac{-k^2}{1+k^2} \le 0, \tag{82}$$

$$M(\rho, k) = -Dk^2 + \frac{\rho k^2}{1+k^2}. \tag{83}$$

The function M attains positive values only for $\rho > D$, which is the critical number identified as $\lim_{\lambda\to\infty} \rho^*(\lambda)$ (14), or the number ρ_0^* in Thm. 1(f) in the case $f(u) = 2uF'(u^2)$, $u_0 > 0$.

All the conclusions of Thms. 2–6 hold (and Thm. 1 as well, if $f(u) = -2uF'(|u|^2)$).

3.8 Related models

The Green's function $G(x)$ used in the previous Section 3.7.1 is just one example of a class of influence functions G in (80) for which our conclusions are valid. We simply require

$$G(x) \geq 0; \quad \int_{-\infty}^{\infty} G(x)dx = 1. \tag{84}$$

Then $Bu = G * u - u$ is a negative self-adjoint operator. If $Au = u_{xx}$, then these two operators satisfy our assumptions throughout, and our conclusions hold regarding patterns for the equation

$$u_t = u_{xx} - \rho(G * u - u) - f(u). \tag{85}$$

A complementary model is obtained when $\rho < 0$. Although such an equation is still a gradient flow, when f is bistable it has traveling wave solutions defined on the whole line. In [8], The existence, uniqueness, and some stability results were obtained for such waves, with and without the second derivative term on the right of (85) (but with $\rho < 0$). This paper was nominally for the equation without the x-derivative, but the results clearly hold with it present. If it is absent, an interesting feature of the traveling waves is that their profile may be discontinuous when the velocity is zero.

Finally, we consider another sort of evolution, when A and B are both convolution operators.

Let $G^{\pm}(y)$ be two functions, each satisfying (84). We consider the evolution equation

$$u_t = G^+ * u - u - \rho(G^- * u - u) - f(u). \tag{86}$$

If we set $G = G^+ - \rho G^-$, then this equation takes the form

$$u_t = G * u - Iu - f(u),$$

where $I = \int_{-\infty}^{\infty} G(y)dy$. We therefore have an integrodifferential equation similar to that in [8, 23, 24, 45], but with a kernel which can change sign.

Identifying $Au = G^+ * u - u, \quad Bu = G^- * u - u$, we obtain (1). Also

$$\hat{A}(k) = \hat{G}^+(k) - 1, \quad \hat{B}(k) = \hat{G}^-(k) - 1, \tag{87}$$

Although assumptions **A1** and **A3** are satisfied, **A2** is not; in fact $\frac{\hat{A}(k)}{\hat{B}(k)} \to 1$.

The following theorem applies to a general situation suggested by this example.

We consider two cases (one may apply at one value of $\rho > \rho^*(\lambda)$, and the other at a different value of ρ):

(i) $M(\rho, k_m)$ has a maximum with respect to m at a finite value $m^*(\rho)$.

(ii) $M(\rho, k)$ approaches its supremum with respect to k at $k = \infty$.

Theorem 7 *Assume that* **A1** *and* **A3** *hold, and*

$$\lim_{k \to \infty} \frac{\hat{A}(k)}{\hat{B}(k)} = 1. \tag{88}$$

If ρ is such that Case (i) holds, then the applicable conclusions of Thm. 1 hold for that ρ.

On the other hand if Case (ii) holds, then there is no global minimizer of exponential form for that value of ρ. However there is a minimizing sequence of exact stationary solutions of (12) of exponential form, along which the wavenumbers approach ∞. The energy levels of these solutions approach a finite limit, as do their amplitudes.

The proof is along the lines of the foregoing, and will be omitted. If $M(\rho, k)$ approaches its supremum only at $k = \infty$, there exist infinite sequences $\{k_m\}$ (see (15)) and a_m, defined for sufficiently large m, which satisfy (22). They generate exponential solutions whose amplitudes approach a solution of (22) with $k = \infty$. As $m \to \infty$, they are ever more finely oscillatory. Although they are not minimizers, their energies are ever closer to the infimum of $E[u]$, so that in that sense their instabilities are ever weaker.

4 Systems with an imposed conservation principle

In place of (8), we now consider the evolution law

$$\frac{\partial u}{\partial t} = \nabla^2 \frac{\delta E[u]}{\delta u}, \tag{89}$$

again for the time evolution of λ-periodic functions of $x \in \mathbf{R}^N$. This equation is still relaxational with respect to E, because along solution paths $u(\cdot, t)$ the following holds, where the scalar products and norms are for $L^2(\Lambda)$:

$$\frac{d}{dt}E[u(\cdot,t)] = \left\langle \frac{\delta E[u]}{\delta u}, \frac{\partial u}{\partial t} \right\rangle = \left\langle \frac{\delta E[u]}{\delta u}, \nabla^2 \frac{\delta E[u]}{\delta u} \right\rangle = -\left\| \nabla \frac{\delta E[u]}{\delta u} \right\|^2 \le 0.$$

Although there are many others, this is the most frequently encountered type of relaxation law with the property that the average value

$$\bar{u} = \frac{1}{|\Lambda|} \int_\Lambda u(x,t)\, dx \tag{90}$$

is independent of t. It represents a gradient flow, not in L^2 (as (8)), but with respect to another scalar product $\langle \cdot, \cdot, \rangle$. One considers periodic functions $w(x)$ with $\bar{w} = 0$, solves $\nabla^2 \Phi = w$ for a Λ-periodic potential Φ with $\bar{\Phi} = 0$. The mapping $w \to \Phi$ is unique. The scalar product is then $\langle w_1, w_2, \rangle_X = \langle \nabla \Phi_1, \nabla \Phi_2 \rangle_{L^2}$. This is in fact can be taken as the scalar product for the dual space $(\bar{H}^1)'$, where $\bar{H}^1$ is the subspace of H^1 consisting of functions with zero average.

For our familiar energy (38), we have the evolution

$$u_t = -\nabla^2 [Au - \rho Bu - f(u)]. \tag{91}$$

Again, we consider solutions which are periodic in space with period cell Λ. Because of the generality of (91), we avoid giving any specific conditions for existence of solutions, and assume that A, B, H are such that the initial value problem is globally well posed, possibly in a weak sense, in some subspace Y of $L^2(\Lambda)$ (i.e. periodic with period cell Λ) containing the domain of E (38).

4.1 Stable patterns

In view of the constancy of $\bar{u}$ (90), it is appropriate to look for a global minimizer of E (9) or (38) under the restriction that its average is a prescribed number $\bar{u} = b$ (say). These solutions satisfy

$$Au - \rho Bu - f(u) - \sigma = 0 \tag{92}$$

for some constant σ.

If $b = 0$, the functional E is as in (9), and we allow complex-valued functions, then necessarily $\sigma = 0$ and the global minimizers studied in subsection 2.2.1 are the same as those in the present context.

On the other hand if we restrict to real-valued functions, then we have a result analogous to Thm. 2, under the additional assumption analogous to (42), but with $\bar{u} = b$ prescribed. For fixed b, set

$$M_b^*(|m|) = \inf_a \frac{H^*(a, b, |m|)}{a^2} \tag{93}$$

and

$$\bar{\rho}_b = \inf \{\rho : M(\rho, k^m) > M_b^*(|m|) \text{ for some } m\}. \tag{94}$$

Theorem 8 *There exists a number $\rho_c(b, \lambda) \in [\rho^*(\lambda), \bar{\rho}_b(\lambda)]$ such that*

a. *for each $\rho < \rho_c$, there exists no nontrivial global minimizer of the function E (38) in the class of real-valued functions satisfying $\bar{u} = b$;*

b. *for each $\rho > \rho_c$, there exists such a restricted nontrivial real global minimizer ϕ^ρ of E in $\mathcal{D}_r \cap L_\infty$ with*

$$E[\phi^\rho] < 0. \tag{95}$$

The proof is similar to that of Thm. 2.

4.2 Metastable patterns: the Cahn-Hilliard equation and another kind of competition

In the case of the Cahn-Hilliard and similar equations, it is appropriate to take $Bu = -u$, so that the destabilizing influence is

$$-\rho Bu = \rho u \tag{96}$$

and $M(\rho, k) = \hat{A}(k) + \rho$. In this case the maximum of $M(\rho, k)$ with respect to k is always attained, like $\hat{A}$, at $k = 0$, and the globally stable states have maximal wavelength. This contrasts with the minimizers in Sec. 2.2, for which the wavelength was never maximal for large enough ρ. The difference is that this choice of B violates part of our assumption **A3**. Nevertheless the most essential parts of the foregoing theory still apply.

As an important example, take the quartic nonlinearity $H(u) = \frac{1}{4}(u^4 + \rho^2)$. Then we may write the last two terms of the energy (38) as

$$\frac{1}{2}\rho\langle Bu, u\rangle + \frac{1}{|\Lambda|}\int_\Lambda H(u)\,dx = \frac{1}{|\Lambda|}\int_\Lambda \left(-\frac{1}{2}\rho u^2 + \frac{1}{4}u^4 + \frac{1}{4}\rho^2\right) dx. \tag{97}$$

If $Au = \nabla^2 u$, we then obtain from (91) the Cahn-Hilliard equation

$$u_t = -\nabla^2(\nabla^2 u + \rho u - u^3). \tag{98}$$

This equation was proposed in [17, 19] as a model for some important features of solidifying molten binary alloys, such as spinodal decomposition. One interprets u as the relative concentration of one component of the alloy. For (98), global existence theories are available; for example this equation is included in a class treated by Henry [51].

Because the global minimizers constrained by $\bar{u} = b$ have maximal wavelength, even for large λ, they do not always provide interesting patterns. Nor are these patterns the most relevant in typical applications. The reason is that a second kind of competition arises in the conserved case, which may result in temporary but long-lasting (metastable) patterned solutions.

To examine this competition, we fix $\rho = 1$ in the above example (97), (98), and in this section consider the problem in one space dimension, with $\lambda = 1$. Then the minimization of the last two terms of E, i.e. $\frac{1}{2}\rho\langle Bu, u\rangle + \int_0^1 H(u)dx = \int \left(H(u) - \frac{1}{2}u^2\right) dx = \frac{1}{4}\int (u^2-1)^2 dx$, is achieved when u is ± 1. To minimize $\frac{1}{2}\rho\langle Bu, u\rangle + \int_0^1 H(u)dx$, therefore, one must require the function u to take the value 1 on one set in $[0, 1]$ and -1 on the remainder of that domain. This is often called "phase separation". Since the total mass $\int u\,dx$ is prescribed, the measures of these sets are chosen to satisfy this constraint.

However, we must also consider the additional term $-\frac{1}{2}\langle Au, u\rangle = \frac{1}{2}\int_0^1 u_x^2 dx$ in the energy (38). It should be as small as possible; this dictates that the transitions between ± 1 should be smooth rather than abrupt, and that there should be as few of them as possible. In other words, the desire to minimize the energy drives the configuration toward longer wavelengths (but they must all fit into the given period interval, which we are taking in this example to be (0,1)). This is the first of our new pair of competing influences.

The second of these influences involves the kinetics of diffusion. It arises because to effect this separation with mass conserved, a redistribution of the particles whose concentration is u will be necessary. This must be accomplished by diffusion (diffusion is provided for by the fourth order term of (98) except when u is near ± 1; then the diffusion is governed by the other two terms on the right). Diffusion takes time, and the time required is longest when the sizes of the regions where $u \approx 1$ or $u \approx -1$ are largest. The competition is therefore between (1) the fact that the energy is least when the length scale of the separation is largest, and (2) the fact that the time it takes to do that increases with this same length scale.

This competition results in a compromise: there is a preferred range of characteristic lengths for the patterning mechanism which is neither too small nor too large. Here the terminology "preferred" takes the following dynamical meaning. Suppose that the initial state $u(x, 0)$ is a small random perturbation of the constant solution $u \equiv \mu$ and that that constant state is unstable. Then a whole range of patterned modes, whose amplitudes start small, will grow. Each mode has its own characteristic spacing. Those modes which have the fastest rates of growth will dictate the characteristic length scale of the emerging pattern. This is the length scale which results from the competition described above.

The modes of growth from those almost constant initial data can be determined from a linearized stability analysis around the constant solution $u \equiv \mu$ (see e.g. [49, 7]). This linear analysis provides a dispersion relation between the growth rate of each mode and the wavenumber k characterizing that mode.

The first rigorous analysis of this part of the separation process was given by Grant [49] for the 1D case. In this stage, we may take "separation" to be synonymous with "patterning". He proved roughly that if the initial state is a sufficiently small random perturbation of the constant state, then with high probability the evolution drives the system to a neighborhood of a stationary periodic exact solution with wavelength predicted by the linear dispersion relation. Such periodic solutions are unstable for λ large enough, and the solution will not remain in that neighborhood forever.

This temporary patterning process is the mathematical analog of spinodal decomposition [18], an event common in the processing of alloys. The subsequent evolution of the system away from the unstable separated solution takes place more slowly. It is also the mathematical analog, in this model, of an important physical process, namely the coarsening of alloys. An extensive study involving both the modeling and numerical simulation of the statistics of coarsening in 1D was undertaken by Eyre [36].

Modeling more complicated alloys leads to systems of Cahn-Hilliard equations, which have even more interesting separation dynamics. Among the notable mathematical studies of these models, I mention the work of Eyre in [37].

Next, we review some mathematical results on separation in higher dimensions.

4.3 Spinodal decomposition in higher dimensions

The process by which patterns are formed spontaneously under Cahn-Hilliard dynamics is considerably more involved in dimensions greater than 1. One reason is that although in 1D the patterns which are observed have maxima and minima and zeros which are approximately evenly spaced (so that the spacing is approximately equal to a number which we may call the characteristic length of the pattern), no such property takes place in 2 or 3 (say) dimensions. The zero set of a solution does not consist of approximately evenly spaced lines, which might be the analog of the 1D situation. Instead, they typically are irregular and curved. Yet they do appear to the eye to possess some characteristic length.

To put the phenomenon of spinodal decomposition on a more satisfactory mathematical footing in higher dimensions, therefore, one needs to provide a notion of characteristic length for a given function and somehow to quantify the assertion that most solutions starting near an unstable constant evolve temporarily to a configuration with an identifiable characteristic length.

We shall consider the Cahn-Hilliard equation (98) with $\rho = 1$ and a small parameter ϵ inserted as follows:

$$u_t = -\nabla^2(\epsilon^2\nabla^2 u - f(u)), \tag{99}$$

Where f is the cubic function indicated in (98), or more generally a function with the same qualitative properties. Then the characteristic length referred to turns out to be $O(\epsilon)$. The equation (99) corresponds to the Ginzburg-Landau energy

$$E[u] = \frac{1}{|\Lambda|}\int_\Lambda \left[\frac{1}{2}\epsilon^2|\nabla u|^2 + H(u)\right] dx, \tag{100}$$

where $f(u) = H'(u)$.

The papers by Maier-Paape and Wanner [56, 57] serve to quantify the intuitive picture we have been describing and to prove some form of it. They consider solutions of the Cahn-Hilliard equation which have evolved from a small neighborhood of an unstable constant solution μ, but which have not evolved too far from it. Thus these solutions start in a neighborhood N_ϵ^0 of it in H^2, and are considered before they leave a larger neighborhood N_ϵ. In 3-dimensional space, both of these neighborhoods have size of order ϵ^3. In other dimensions, this exponent 3 here and below should be replaced by the dimension of the space.

To develop a reasonable assertion, one first considers the eigenvalues of the Cahn-Hilliard operator linearized about μ, and collects in a group those which are close to the maximal one, which is positive. The number of these eigenvalues is strongly ϵ-dependent and will be very large when ϵ is small. They generate a strongly unstable finite dimensional invariant manifold M_ϵ for the flow, which is imbedded within another finite dimensional inertial manifold M_ϵ' with much larger dimension. The crucial statements are that solutions starting in N_ϵ^0 are exponentially attracted to M_ϵ', and that most of them exit N_ϵ being close to M_ϵ. The meaning of "close" and "most" is this: one can choose an arbitrarily small number δ_ϵ proportional to ϵ^3 and a probability p arbitrarily close to 1 and conclude that if the diameter of N_ϵ^0 is taken small enough relative to ϵ^3 and initial conditions are taken with projections onto the tangent space of M'_ϵ in a random manner, then the probability of the orbit exiting N_ϵ at a point closer than δ_ϵ to M_ϵ is at least p.

The second conceptual result is that those orbits which are close to M_ϵ have a characteristic length which is $O(\epsilon)$. This idea needs to be clarified. They propose tying it to an upper bound for the distance between adjacent portions of the nullset for the function u, which we can think of as being the solution of the CH equation at the point in time of its exit from N_ϵ. This means that one looks at a connected set Ω in space bounded by a set where u vanishes. Thus $u > 0$ or $u < 0$ on Ω. One takes a ball of radius r contained in Ω and asks how large r can be. An upper bound, given in terms of ϵ, for such radii can be established for most of the solutions mentioned above. This upper bound is identified as the characteristic length of the pattern.

First of all, this upper bound is proved for functions which are a linear combination of those eigenfunctions of the linearized CH operator corresponding to the chosen eigenvalues close to the maximal one. A technical restriction is placed on the center of the ball. Since at the exit time most of the solutions are close to the invariant manifold M_ϵ, which is in turn close to the span of these eigenfunctions, the property holds for them as well.

In short, these authors have proved that with high probability, solutions exhibit a mathematical phenomenon like spinodal decomposition, a type of phase separation, by the time they leave a ball of radius $O(\epsilon^3)$ in H^2 centered at μ, provided that they begin randomly in a smaller neighborhood with radius of the same order of magnitude. Numerical simulations (of which there are many; see for example [34]) show that separation with characteristic length persists outside of that small neighborhood, when in fact the amplitude of the separated solution is $O(1)$.

Extensive Monte-Carlo simulations in 1D were performed by Sander and Wanner [79] in order to gain insight into the question of how long the decomposition persists. They made a systematic study of the distribution of exit points on the boundaries of balls V_ϵ of various radii centered at μ. Their observations indeed confirm the persistence of the separation mechanism beyond the confines of the small neighborhood figuring in the proofs in [56, 57]. In fact they indicate that the solutions of the Cahn-Hilliard equation remain close to the solutions of the equation linearized about the constant $u = \mu$ much longer than one would expect—far into the nonlinear regime. This is important since the solutions of the linearized problem are built up from the various linear modes growing according to their known growth rates, and it is easily seen that the fastest growing ones will eventually dominate. As mentioned above, this domination is another way of expressing the spinodal decomposition event.

The unexpected validity of the linear approximation for solutions growing to larger amplitudes can be given a mathematical justification. The following argument is based on that given in [79]; this and more have been proved in a forthcoming publication..

Consider (99) with $f(u) = -u + u^3$, and $\mu = 0$. We may decompose the operator on the right as

$$-\Delta(\epsilon^2\Delta + 1)u + \Delta(u^3) = L_1 u + L_2 u. \tag{101}$$

Let u be the solution with $u(x, 0) = \delta\phi(x)$, where $\delta \ll 1$ and ϕ is an eigenfunction of the linearized operator L_1 corresponding to one of the group of fastest growing modes. For those modes, it can be calculated that the eigenvalues κ of L_1 are $O(\epsilon^{-2})$, and that $\|L_1 u\|_{L^2} \le C\epsilon^{-2}\|u\|_{L^2}$.

During this time, it is expected that in most cases the L^∞ norm of u will be bounded by unity; the largeness of $L_1 u$ is due to its large derivatives.

On the other hand, $L_2 u$ can be estimated to be bounded in the form $\|L_2 u\|_{L^2} \le C\epsilon^2\|u\|_{H^2}^2\|u\|_{L^2}$, as long as u remains close to the linear mode, due to the fact that ϕ is an eigenfunction of the Laplace operator with eigenvalue $O(1)$. Therefore the magnitude of the nonlinear part $\|L_2 u\|_{L^2}$

relative to that of the linear part $\|L_1 u\|_{L^2}$ will be of the order of $\epsilon^4\|u\|^2_{H^2}$. This allows $\|u\|_{H^2}$ to grow quite large before the nonlinear terms become significant.

The conclusion is that the nonlinearity is relatively insignificant during the time the solutions grow to moderate size.

This heuristic result in 1D of Sander and Wanner has actually been proved and improved by them, and extended to dimensions 2 and 3 in a forthcoming paper. They have shown that similar estimates for the nonlinear versus linear part of the solution are satisfied not just for solutions starting as $\delta\phi(x)$, but in fact on the whole dominating subspace, i.e. for solutions u which start as superpositions of several dominating eigenfunctions, and in fact for those which start "close" to that subspace.

4.4 Interfaces

The previous section was about patterning in solutions of the Cahn-Hilliard equation, in the sense of the spontaneous formation of nonconstant solutions whose spatial configurations are dominated by structures with characteristic lengths. The configurations are not stable stationary states, however. In time, numerical simulations indicate that other structures appear with separated length scales: spatial domains appear within which the solution $u(x,t)$ of (99) has a degree of smoothness which is independent of ϵ, and in each of which u assumes values near one of the stable zeros of the function $f(u)$. These domains are separated by narrow internal layers of characteristic width $O(\epsilon)$. Within these layers, the solution makes a transition between neighborhoods of the two stable zeros of f, namely the ones belonging to the two domains on either side of the layer.

At this stage in the dynamics of (99), the characteristic length of the pattern has increased from $O(\epsilon)$ to $O(1)$. Thus coarsening has progressed considerably. In terms of the competition discussed in Sec. 4.2, this shows that in time the first influence, namely the fact that the energy E is smaller at larger characteristic lengths, wins out.

The properties of these layers, especially their dynamics, have been studied to a considerable extent. We shall briefly review the results. First, we recount results for the nonconserved evolution analogous to (99), namely a form of the Ginzburg-Landau equation studied in a materials science context by Allen and Cahn:

$$u_t = \epsilon^2\nabla^2 u - f(u). \tag{102}$$

This is the nonconserved equation $\frac{\partial u}{\partial t} = -\frac{\delta E}{\delta u}$ sharing the same Ginzburg-Landau energy (100).

4.4.1 Ginzburg-Landau-Allen-Cahn interfaces in 1D

The slow dynamics of layered solutions of (102) was discovered by J. Neu (unpublished), when the function f is of "balanced bistable" type. For simplicity, we take the stable zeros of f to be at $u = \pm 1$. Then "balanced bistable" in the present context will mean that $f'(\pm 1) < 0$, f has only one zero between the two stable ones, and $\int_{-1}^{1} f(u)\,du = 0$. Neu considered the problem

$$u_t = \epsilon^2 u_{xx} - f(u), \quad x \in (0,1), \quad u_x = 0, \quad x = 0, 1. \tag{103}$$

Throughout the following, we assume that $0 < \epsilon \ll 1$.

There are solutions such that $u(x,t) = 0$ for $x = \xi(t)$ for some time interval, $u(x,t)$ is exponentially (in ϵ) close to 1 for $x > \xi(t) + \delta$, and to -1 for $x < \xi(t) - \delta$, for any $\delta > 0$ independent of ϵ. Moreover, the following order of magnitude relation holds for the motion of the interface:

$$\xi'(t) \simeq C\epsilon\left(e^{-(1-\xi(t))/\epsilon} - e^{-(\xi(t))/\epsilon}\right). \tag{104}$$

Thus, the motion is exponentially slow and is directed toward the nearest boundary. Moreover, the velocity increases dramatically when the distance to that boundary is decreased by an $O(1)$ amount.

This result was generalized to the multilayer case and established rigorously by Carr and Pego [20] and Fusco and Hale [46]. The asymptotic analysis of this phenomenon was further developed in considerable detail by Reyna and Ward [83, 75].

There are many stationary solutions with layered structure at equally spaced locations on the interval (0,1). During the period of time when the slow motion of the interfaces described above persists, the solution stays close to the unstable manifold of one of these layered states. This remains true until two layers are close enough to each other, or one of them is close enough to the boundary. At that time, the velocity of these layers will have reached the order of magnitude $O(1)$, and the two layers quickly annihilate each other (or one is annihilated at the boundary).

This event decreases the number of layers, and shifts the solution to a different unstable manifold.

Key features in the rigorous justification of parts of this picture are the construction of good approximations to the unstable manifolds in terms of coordinates that correspond to physical space, and careful estimation of the exponentially small eigenvalues of the linearization of the differential operator about the layered solution.

In the same spirit as the above, a problem in higher dimensions related to the perturbation of a layered equilibrium state by changes in the shape of the domain was considered by Alikakos, Fusco, and Kowalczyk [6].

4.4.2 Cahn-Hilliard slow motion in 1D

Similar results hold for the Cahn-Hilliard equation (99) in one space dimension:

$$u_t = -(\epsilon^2 u_{xx} - f(u))_{xx}, \quad u_x = u_{xxx} = 0 \text{ for } x = 0,\, 1. \tag{105}$$

Because this equation conserves "mass" $\int_0^1 u(x,t)\,dx$, the dynamics of the layers are different, and in fact when there is only one of them, it does not move.

The extension of the results of Fusco-Hale-Carr-Pego to this equation in the case of two layers was done in the paper of Alikakos, Bates, and Fusco [1]. Later, this was done for more layers by Bates and Xun [9, 10]. Again, the careful construction of approximate unstable manifolds was a key feature of the proof. Some of the spectral estimates used the results of [7].

4.4.3 Cahn-Hilliard slow motion in higher dimensions

Consider now solutions of (99) in higher dimensions. The analog of the layers discussed above are curves (in 2D) or surfaces which are either closed or intersect the boundary. Again, these curves separate domains in which u takes values near one of the two stable zeros of f. When they are very close to being circular, the motion is again exponentially slow.

The study of solutions structured in this manner relies, as before, on careful estimates of the spectrum of the operator linearized about a layered function. This was initiated by Alikakos and Fusco [3], who did the analysis in 2D; it was followed by Chen [22] in higher dimensions.

In [4], the spectra of circular or spherical solutions with a single layer (which they term a "bubble") was studied, and exponentially small eigenvalues revealed in any dimension. In [5], these authors established the exponentially slow motion of layered solutions which approximate a single bubble, in higher dimensions. (If more than one bubble are present with different diameters, the motion speeds up; the larger one will grow while the other decreases, at a rate faster than exponentially slow. Thus motion with more than one bubble is inherently unstable.) One of the new features of this paper is the development of a method for determining the equilibrium positions of the bubbles.

In this paper, the authors use what they term the "manifold approach". Consider "bubble" solutions u^ξ of fixed radius centered at the variable point ξ. As ξ varies, this family of functions forms an approximate invariant manifold for the evolution (99). That equation says that motion at a point u on an invariant manifold is in the direction $\mathcal{L}u = -\nabla^2(\epsilon^2\nabla^2 u - f(u))$, so that $\mathcal{L}u$ is tangent to the manifold. On the other hand, the derivatives $\frac{\partial u^\xi}{\partial \xi_i}$ form a basis for the tangent space to the above-mentioned approximate invariant manifold at the point u^ξ. We therefore have the approximate relation

$$\mathcal{L}(u^\xi) = c_i(\xi)\frac{\partial u^\xi}{\partial \xi_i}$$

for some velocity functions $c_i(\xi)$, for which estimates can be obtained. This is a kind of global Lyapunov-Schmidt reduction to a finite-dimensonal dynamical system. The equilibrium position of a bubble corresponds to the condition $c(\xi) = 0$. Despite the clear geometric picture which emerges, the proof of its validity is extremely involved.

In [14], Bronsard and Kohn gave a simple energy argument establishing that a layer for the 1D Allen-Cahn equation takes superalgebraically long to move a distance of the order $O(1)$. Their method was refined later by C. Grant [50] to give exponentially long time behavior, also for systems of equations. However, the Bronsard-Kohn method does not establish "exponentially slow motion until two layers get close", and persistence of shape until they get close. These results were strengthened in [2], in which the energy approach of [14] together with the spectral facts of [4] were used to establish in a relatively simple manner slow motion in higher dimensions as well until "two layers get close together, or the bubble gets close to the boundary". The difference between this result and that of [5] is that the vector field now is only estimated, not determined precisely enough so that equilibrium can be determined.

4.4.4 The role of exponential asymptotics

We have given a brief outline of the phenomenon of exponentially slow dynamics for the Allen-Cahn and Cahn-Hilliard equations, together with associated analytical results. The details of this dynamics can be discovered by formal asymptotics based on the smallness of ϵ, but this procedure is not standard, in the sense that exponentially small quantities must be accounted for.

Considerable effort has been expended to construct exponential asymptotic theories for "layered" and "spike" solutions in these and several other contexts, including viscous shocks, the viscous Cahn-Hilliard equation, and the Gierer-Meinhardt system. We refer the reader to the surveys found in [84, 85, 86].

4.5 Phase-field models

Our discussion of pattern-forming conserved evolutions would not be complete without mentioning the increasingly popular diverse class of phase-field models. These models are used to investigate, both numerically and theoretically, the properties of materials which undergo phase transformations, especially the dynamical and structural properties of the interfaces between two phases.

The phenomenology of these models parallels that of the Cahn-Hilliard equation. In fact the mathematical spinodal decomposition event occurs ([41, 42]), and interfaces form and move. The spinodal decomposition phenomenon appears not to closely mirror any important laboratory observation. As for the interfaces, the analogy with the Cahn-Hilliard model breaks down here, because interfacial motion is not in general slow. Rather it is typically governed by some known free boundary problem like a modified Stefan problem. Examples of formal reductions to such problems can be found in [16, 15, 44, 21].

The following is a representative but certainly not universal setting for these models.

One begins with an order or phase parameter function $\phi(x,t)$ which is meant to represent the phase of a material that can exist in two phases, except that it is a smooth function with range an interval on the real line. Pure phases are represented by extreme values of ϕ, and other values represent the state of the material at locations where it may be in the process of changing phase. One also has an internal energy density function e and an entropy density, which is a given function $s = s(e, \phi, \nabla\phi)$ of e, ϕ and the spatial gradient of ϕ.

We consider constrained gradient flows for the total entropy

$$S[e,\phi] = \int s(e,\phi,\nabla\phi)dx, \tag{106}$$

where the integral is over the spatial domain occupied by the material in question. The constraint is that the total internal energy

$$E(t) = \int e(x,t)\, dx \tag{107}$$

be constant.

The corresponding gradient flow for the functions $e(x,t)$, $\phi(x,t)$ is:

$$\begin{array}{rcl} \frac{\partial e}{\partial t} &=& -\nabla D \nabla \frac{\delta S}{\delta e}, \\ \frac{\partial \phi}{\partial t} &=& K \frac{\delta S}{\delta \phi}. \end{array} \tag{108}$$

Here D and K are positive functions.

In (108), the second equation would be the usual gradient flow for the functional $-S$ if the variable e were not present. The first equation would be a constrained gradient flow analogous to (89) if there were no other variable ϕ. If both equations hold, then it is easy to check that $\frac{d}{dt}S[e(\cdot,t),\phi(\cdot,t)] \geq 0$, so that $-S$ is a Lyapunov functional and global minimizers are stable.

In the simplest cases, $\frac{\delta S}{\delta e} = \frac{1}{T}$ where T is the temperature of the material, so that with proper choice of D, $(108)_1$ can be made into the usual heat conduction equation, in which the flux of internal energy e is proportional ∇T. In that case, one should express e as a function $e(T,\phi,\nabla\phi)$ in a way that is thermodynamically compatible with the function $s(e,\phi,\nabla\phi)$; the system then becomes one for the evolution of T and ϕ. Also in the simplest cases, $\frac{\delta S}{\delta \phi}$ is like the Ginzburg-Landau bistable operator on the right of (103) and $(108)_2$ assumes the form

$$\frac{\partial \phi}{\partial t} = \frac{K}{T}[\epsilon^2 \nabla^2 \phi - f(\phi, T)], \tag{109}$$

$\epsilon \ll 1$, where the function f, in its dependence on ϕ, is bistable and grows fast enough as $\phi \to \infty$, as we assumed of the function f in (12).

I have sketched the derivation of typical simple phase-field equations on the basis of a principle that the total entropy in the system should increase. The postulate in doing this is that the evolution should be the corresponding gradient flow. However, a local entropy principle can also be used to produce the same result without this second postulate. See, for example, [21]. In its place, one postulates certain phenomenological relations designed so that the local entropy production is nonnegative.

5 Patterns and phase evolutions

Patterns whose amplitude and wavelength vary slowly and on long spatial scales are seen repeatedly in experiments and numerical simulations. Multiscale perturbation methods, going back at least to the work of Eckhaus [33] and Benjamin and Feir [11], have been useful in deriving approximate evolution laws which govern these modulated solutions of nonlinear PDE's.

For example, small amplitude patterns arising when the control parameter is just past its critical value have dynamics given by amplitude equations [68]. Amplitude equations are derived formally by a weakly nonlinear analysis of the equations with the ansatz that the solutions can be expressed as functions not only of the original space and time variables but also of long range ones. If ϵ is the ratio of small to long space scales, the solution is then expanded formally in powers of ϵ, up to a few orders. This expanded form of the desired solution is put into the original equation and each order considered separately to derive, with the help of solvability conditions belonging to singular operators, equations for the various terms in the expansion. What results is typically a lowest order equation for the evolution of the amplitude in the long space and time variables which is much simpler than the original PDE.

This simplicity is of course paid for by the restrictive assumptions on the form of the solution. Nevertheless, instabilities in the original unmodulated regular pattern are found this way, and it is widely believed that these modulations are an accurate reflection of nature.

Larger amplitude patterns in higher dimensions for values of the control parameter well past criticality are also capable of modulations on such long space-time scales. When these basic patterns are rolls, i.e. periodic in one spatial direction and constant in others (and in the case of traveling waves also periodic in time), one sees slow modulations in the wavelength, orientation and amplitude of the solutions, of which the first two are the most important. These modulations may result in rolls which are now only approximately straight, or whose wavelength is only approximately constant, in local regions of space. The mathematical formalism for this scenario was developed in perhaps the greatest generality by Cross and Newell [32]. Here we briefly describe their procedure.

First, consider a pattern of unmodulated rolls. The state u of the system at a given point in space and time can be specified exactly in terms of the value of the phase θ of the (spatial and/or temporal) oscillation, θ living on the unit circle. With no loss of generality, we may take θ to be $\theta(x,t) = k{\cdot}x - \omega t \,(\operatorname{mod} 2\pi)$ for some constant vector wavenumber $k = \nabla\theta$ pointing in the direction orthogonal to the rolls. Since we shall deal only with derivatives of θ, we shall ignore the "mod 2π" provision. The solution is then given by a periodic function $u = u(\theta)$. For simplicity we consider only the case $\omega = 0$.

For modulations, the wave-form $u(\theta)$ will vary slowly in space and time, and the wavenumber k will do so as well. The first natural ansatz in this case is therefore that the field function $u = u(\theta, X, T)$, which locally is a function only of θ, is also a function of the long space and time variables X and T. As before, let $\epsilon \ll 1$ be the ratio of short to long space scales: $X = \epsilon x$; it may be taken to be arbitrary.

The second ansatz is designed to express the modulation of k, while retaining its meaning as the spatial gradient of θ. In the unmodulated case with $k =$ const., we have that $\epsilon\theta(x) = \epsilon k{\cdot}x = k{\cdot}X \equiv \Theta(X)$ is a function only of X. In the modulated case, one still assumes that the long-scale phase Θ defined as $\Theta = \epsilon\theta$ is a function only of X and T (it may also depend on ϵ). The slowly varying wavenumber is then given by $k(X,T) = \nabla_X \Theta(X,T)$.

When the assumed form $u = u(\theta, X, T)$ is substituted into the basic equations, one identifies $\nabla_x \theta = \nabla_X \Theta$, so that

$$\nabla_x u = k{\cdot}u_\theta + \epsilon \nabla_X u = \nabla_X \Theta {\cdot} u_\theta + \epsilon \nabla_X u.$$

Similar expressions are used for time derivatives.

Then one proceeds as before to expand u and Θ in low powers of ϵ, substitute these expansions into the original PDE's rewritten for functions of the long-range as well as the original space/time variables, and examine each order (power of ϵ) separately. The resulting equations are again analyzed with the use of solvability conditions.

The Cross-Newell equation is an equation for the field $\Theta(X,T)$, and is derived as indicated. It is usually written in the form

$$\Theta_T - M_1(k)\nabla\cdot\hat{k} - M_2(k)(\hat{k}\cdot\nabla)k = 0, \tag{110}$$

where $\hat{k}$ is the unit vector in the direction of k. Since $k = \nabla_X\Theta$, this equation can be written as a second order equation for Θ of the form

$$\Theta_T - \sum_{i,j} M_{i,j}(\nabla\Theta)\frac{\partial^2\Theta}{\partial X_i \partial X_j} = 0 \tag{111}$$

for some nonlinear functions $M_{i,j}$ depending on the original field equations and the original roll solutions. Under some special conditions, this equation takes on a divergence form, so that it is a gradient flow for an energy functional whose integrand is a nonlinear function of k. This condition makes the analysis easier.

In typical cases, when the original periodic roll solution is stable according to the linear stability criterion, the equation (111) is parabolic, so that its initial value problem is well posed for at least a short time interval.

There has been work directed to using the Cross-Newell equation to explain pattern defects such as disclinations (places where the vector k suddenly changes direction or is ill-defined, dislocations (where a roll terminates) or grain boundaries (boundaries between regions with different pattern orientations). It is surmised that near defects, the local wavenumber k is forced outside the domain where the corresponding roll would be stable. At these locations, (110) loses its well-posedness, and therefore its physical relevance, unless other interpretive considerations are added to it.

However, a weak solution or a regularized solution of (110) can in many cases be defined. The singularities of weak solutions and analogous properties of regularized ones are then interpreted as defects, including domain boundaries between differently oriented regular patterns. See [73, 67, 66] for explorations of these issues. In particular, [66] contains a thorough analysis of the Cross-Newell equation, in part from a geometric point of view.

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