A convolution model for interfacial motion: the generation and propagation of internal layers in higher space dimensions.

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Abstract

Properties of solutions of a bistable nonlinear convolution equation in higher space dimensions are studied. The nonlinearity is the derivative of a double well function. The theory of traveling waves for this equation was given in a previous publication [4]. Here we consider spreading phenomena for state regions, in some cases by means of the motion of domain walls, which are modeled by internal layers. These phenomena are analogous to those known for the bistable nonlinear diffusion equation, and in particular, for the Allen-Cahn equation, which is a model for the motion of some grain boundaries in solid materials. Cases when the two wells have unequal depth are considered, as well as when they have equal depth. In the latter case a motion-by-curvature law is derived formally in two space dimensions.

1 Introduction

We consider the properties of solutions $u(x,t)$ of the bistable nonlinear convolution equation

$$u_t = J * u - u - f(u), \qquad x \in R^N, \quad t \geq 0, \tag{1}$$

in which the function f has "stable" zeros at $u = \pm 1$ and the kernel J of the convolution term is nonnegative with unit integral. This equation was introduced in [4] as an L_2 gradient flow for the energy functional

$$E[u] = \frac{1}{4}\int\int J(x-x')(u(x)-u(x'))^2 dx dx' + \int W(u(x))dx. \quad (2)$$

Here the function $W(u)$ is a double well function with $f(u) = W'(u)$, and the integrals are over R^N. This evolution equation (1) is, as a gradient flow [13, 14], analogous to the more familiar bistable nonlinear diffusion equation

$$u_t = \Delta u - f(u), \quad (3)$$

for which the energy is the Ginzburg-Landau functional

$$E_{GL}[u] = \int \frac{1}{2}|\nabla u(x)|^2 dx + \int W(u(x))dx. \quad (4)$$

In the case that W is balanced, i.e. $\int_{-1}^{1} f(u)du = 0$, the equation (3) is referred to as the Allen-Cahn equation [6, 7, 8].

The focus of attention in [4] was the existence and properties of traveling waves of (1) in one space dimension. The most important kind of solutions of (1) are those for which at every instant of time, the solution u takes values near $+1$ in some region of space and values near -1 in another region. We shall refer to these as state regions, analogous to regions occupied by different solid grains modeled by (3). In the case of planar traveling waves, whether considered in one or more space dimensions, these regions are half-spaces. Depending on whether the velocity is positive or negative, traveling waves with nonzero velocity represent the progressive spread or contraction of the region occupied by the "upper" state (for which $u \approx 1$).

In the present paper also, we are concerned with questions of the spread or contraction of regions occupied by the upper or lower state, but not necessarily in the form of traveling waves. Thus, those regions in our paper will be regions in R^N which may have irregular and sometimes ill-defined boundary. In some cases, the interfacial boundary between the two state regions will be a narrow strip or sheet, referred to as an internal layer. One of our main interests will be in the process by which such an internal layer may arise spontaneously (Section 2) and may migrate.

The speed of propagation of state regions is studied in general terms in Sec. 3, and specifically in the case when the boundary is a thin layer in Sec.

4. The rate of spread of state regions is shown to be related to the speed of the corresponding 1D traveling waves when the latter is nonzero. When this latter speed vanishes, then the motion of higher dimensional layers will be slower by a factor proportional to the thickness of the layer, and will also be approximately proportional to the curvature of the layer. This is shown by a nonrigorous asymptotic analysis in Sec. 5.

All these properties are known to occur as well for the bistable nonlinear diffusion equation (3), and have been studied extensively ([1, 3, 5, 9, 10, 12, 15, 16, 17, 11, 19, 24, 25, 26, 28, 29, 30]).

Another nonlinear convolution model [21, 22, 23, 20, 27], arising from the dynamic Ising model, has also been investigated very intensively along some of these lines. In particular, it also has solutions with layers which were proved in [21] to be driven by their mean curvature.

Most of our results are proved for the case that J is isotropic, i.e. depends only on the distance between x and x'. However, anisotropy of J is allowed in Sec. 2 and Sec. 5, where its implication for the motion by curvature law is a prime consideration. The anisotropy may result in this law being ill-posed for certain orientations of the interface. This is a common phenomenon in crystal growth as well.

After the paper was written, we were informed that Barles and Souganidis had just obtained some results intersecting part of our paper; they appear in [2] (see also the survey [31]). The paper [2] treats, in a general setting, curvature laws for the propagation of interfaces in any space dimension higher than 1 as in our sec. 5, but under the condition that the traveling wave solution has zero velocity and is C^1 smooth. However, the initial generation of interfaces is not proved in [2], nor the propagation of interfaces when the 1-D traveling wave has nonzero velocity; these are both done in this paper. The initial generation result is needed in [2] to prove the curvature law.

2 Initial generation of interfaces

Throughout the paper, we assume

A1: $J \in C^2(R^N)$, $J(-x) = J(x)$, $J(x) \geq 0$, $\int J(x)dx = 1$, and $\int |x|^2 J(x)dx < \infty$.

A2: $f \in C^2(R)$; f has exactly three zeros, namely ± 1 and $\alpha \in (-1,1)$; $f'(\pm 1) > 0$. f has no other zeros. Moreover, $f'(\alpha) < 0$. (This last is needed only in the present section.)

The appearance of layered solutions of the bistable nonlinear diffusion equation (3) is known ([26, 15, 9, 28]) to occur in the "fast reaction, slow diffusion" case, when space and time are scaled with a small constant ϵ so that (3) assumes the form

$$\epsilon u_t = \epsilon^2 \Delta u - f(u). \tag{5}$$

The analogous scaling for (1) yields the following analog of the "small diffusion" term: $J_\epsilon * u - u$, where

$$J_\epsilon(x) = \frac{1}{\epsilon^N} J\left(\frac{x}{\epsilon}\right). \tag{6}$$

Accordingly, the present section considers the initial value problem

$$\begin{aligned} \epsilon u_t &= J_\epsilon * u - u - f(u), \qquad x \in R^N, \quad t \geq 0, && (7)\\ u(x,0;\epsilon) &= \phi(x), \quad -1 \leq \phi(x) \leq 1. && (8) \end{aligned}$$

Here, the initial function ϕ does not depend on ϵ, and is continuous with bounded second derivatives. We denote the solution by $u(x,t;\epsilon)$, and sometimes by $u^\epsilon(x,t)$.

The goal is to show that in the region of space where $\phi(x) \geq \alpha + O(\epsilon^{1/2}|\ln\epsilon|)$, the solution $u(x,t;\epsilon)$ becomes close to the value 1 in time $O(\epsilon|\ln\epsilon|)$, and similarly where $\phi(x) \leq \alpha - O(\epsilon^{1/2}|\ln\epsilon|)$, u becomes close to -1 in the same time. Thus interfaces are generated in a time scale $O(\epsilon|\ln\epsilon|)$ and the "thickness" of the generated layers is at most $O(\epsilon^{1/2}|\ln\epsilon|)$. This can be compared to the result by Chen in [9], in which it is proved that the nonlinear diffusion equation (5) develops (and propagates) layers with thickness $O(\epsilon|\ln\epsilon|)$.

Theorem 1. Suppose that $\|\phi\|_{C^2}$ is finite. Assume A1 and A2. There exist constants $\tau_0 > 0$ (depending only on f) and $M_0 > 0$, $M_1 > 0$, both depending on f and $\|\phi\|_{C^2}$, such that

(i) for small $\epsilon > 0$, if x is such that $\phi(x) \geq \alpha + M_0\sqrt{\epsilon}|\ln\epsilon|$, then $u(x,\tau_0\epsilon|\ln\epsilon|) \geq 1 - M_1\epsilon$, and

(ii) if $\phi(x) \leq \alpha - M_0\sqrt{\epsilon}|\ln\epsilon|$, then $u(x,\tau_0\epsilon|\ln\epsilon|) \leq -1 + M_1\epsilon$.

In our proof, we use the method in [9] with modifications.

We shall prove only part (i) in details; the proof of part (ii) follows by applying (i) to $-u(-x,t)$.

We start by perturbing $f(u)$ near $u = \alpha$ as follows:

$$f_\epsilon(u) = (1-\rho_\epsilon(u))f(u) + \rho_\epsilon(u)\frac{\alpha + \epsilon^{1/2}|\ln\epsilon| - u}{|\ln\epsilon|}, \tag{9}$$

where ρ_ϵ is a nonnegative C_0^∞ cut-off function satisfying

$$\begin{array}{l}\rho_\epsilon(u) \equiv 0 \text{ for } u \leq \alpha - \sqrt{\epsilon}/C_1 \text{ and } u \geq \alpha + 3\sqrt{\epsilon}|\ln \epsilon|,\\ C_1 = \|f'\|_{L^\infty([-1,1])},\\ \rho_\epsilon(u) \equiv 1 \text{ for } \alpha \leq u \leq \alpha + 2\sqrt{\epsilon}|\ln\epsilon|,\\ 0 \leq \rho'_\epsilon(u) \leq 2C_1/\sqrt{\epsilon} \text{ for } u < \alpha,\\ 0 \geq \rho'_\epsilon(u) \geq -2/(\sqrt{\epsilon}|\ln \epsilon|) \text{ for } u > \alpha.\end{array} \tag{10}$$

We list the properties of f_ϵ proved as in [9]:

P1. f_ϵ has exactly three zeros: $-1,\ \alpha + \sqrt{\epsilon}|\ln\epsilon|,\ 1$;

P2. $f_\epsilon(u) \geq f(u)$ for all u in, say, [-2,2];

P3. $|f'_\epsilon(u)| \leq C_2$ for all $u \in [-2,2]$ and all small ϵ (C_2 is a constant);

P4. there exists a positive constant C_3 independent of small ϵ such that

(a) $f_\epsilon(u) \geq C_3\sqrt{\epsilon}$ for $u \in [\alpha - \frac{\sqrt{\epsilon}}{C_1}, \alpha]$,
(b) $f_\epsilon(u) \leq -C_3\sqrt{\epsilon}$ for $u \in [\alpha + 2\sqrt{\epsilon}|\ln\epsilon|,\ \alpha + 3\sqrt{\epsilon}|\ln\epsilon|]$.

Consider the ordinary differential equation obtained by ignoring the "diffusion" term in (7) with $f(u)$ replaced by $f_\epsilon(u)$:

$$\epsilon v_t(\eta,t) = -f_\epsilon(v(\eta,t)), \tag{11}$$
$$v(\eta,0) = \eta \in R^1. \tag{12}$$

For a slightly different f_ϵ, the following lemma was proved in [9], except for (iii):

Lemma 2. (i) There exists a positive constant τ_0 such that for all small ϵ,

$$v(\eta,t) \geq 1 - \epsilon^2 \text{ for } t \geq \tau_0\epsilon|\ln\epsilon|, \quad \eta \geq \alpha + 3\sqrt{\epsilon}|\ln\epsilon|.$$

(ii) $v(\eta,t)$ is C^2 smooth in both η and t, and

$$v_\eta > 0 \text{ for all } \eta \in [-2,2], \quad t \geq 0.$$

(iii) There exists a positive constant C_4 such that for all small ϵ and for $0 \leq t \leq \tau_0\epsilon|\ln\epsilon|$, $\eta \in [-2,2]$,

$$|v_{\eta\eta}(\eta,t)| \leq \frac{C_4}{\epsilon}. \tag{13}$$

Proof of (iii): The proof of this is very close to that of Chen [9] when he proves $\left|\frac{v_{\eta\eta}}{v_\eta}(\eta,t)\right| \le M/\epsilon$.

By [9],

$$v_{\eta\eta}(\eta,t) = \frac{(f'_\epsilon(v(\eta,t)) - f'_\epsilon(\eta))f_\epsilon(v(\eta,t))}{f^2_\epsilon(\eta)}.$$

To show (13), by (P3) we need only worry about those η such that $|f_\epsilon(\eta)| \le C\sqrt{\epsilon}$ for some small constant C. These η are contained in $(-1-\sqrt{\epsilon}, -1+\sqrt{\epsilon})$ or $(1-\sqrt{\epsilon}, 1+\sqrt{\epsilon})$ or $[\alpha, \alpha+2\sqrt{\epsilon}|\ln\epsilon|]$, by (P1), (P3), and (P4).

If $\eta \in (-1-\sqrt{\epsilon}, -1+\sqrt{\epsilon})$, then $v(\eta,t)\downarrow -1$ as $t\to\infty$. Thus

$$\begin{aligned}&|v_{\eta\eta}(\eta,t)| \le \tfrac{|f''(\ldots)||v(\eta,t)-\eta|f(v(\eta,t))}{f^2(\eta)} \\ &\le \tfrac{C_5|-1-\eta||-1-v(\eta,t)|}{|-1-\eta|^2} \le C_5,\end{aligned}$$

where C_5 depends on f and its derivatives up to second order near $u=-1$. Similarly, we can show that $v_{\eta\eta}$ is bounded for $\eta \in (1-\sqrt{\epsilon}, 1+\sqrt{\epsilon}) \cup (-1-\sqrt{\epsilon}, -1+\sqrt{\epsilon})$.

It remains to prove (13) for $\eta \in [\alpha, \alpha+2\sqrt{\epsilon}|\ln\epsilon|]$.

Case 1. $v(\eta,t) \in [\alpha, \alpha+2\sqrt{\epsilon}|\ln\epsilon|]$ for all $t \le \tau_0\epsilon|\ln\epsilon|$.

Then $v_{\eta\eta}(\eta,t) = 0$, since f_ϵ is linear for $u \in [\alpha, \alpha+2\sqrt{\epsilon}|\ln\epsilon|]$.

Case 2. For some $t_0 \le \tau_0\epsilon|\ln\epsilon|$, $v(\eta,t_0) = \alpha+2\sqrt{\epsilon}|\ln\epsilon|$. Then by solving (11), we find

$$\begin{aligned}v(\eta,t) &= \alpha+\sqrt{\epsilon}|\ln\epsilon| + \sqrt{\epsilon}|\ln\epsilon|e^{(t-t_0)/\epsilon|\ln\epsilon|}, \quad t \le t_0, \\ \eta = v(\eta,0) &= \alpha+\sqrt{\epsilon}|\ln\epsilon| + \sqrt{\epsilon}|\ln\epsilon|e^{-t_0/\epsilon|\ln\epsilon|} \ge \alpha+\sqrt{\epsilon}|\ln\epsilon|(1+e^{-\tau_0}), \\ f_\epsilon(\eta) &= \tfrac{\alpha+\sqrt{\epsilon}|\ln\epsilon|-\eta}{|\ln\epsilon|} \le -\sqrt{\epsilon}e^{-\tau_0}.\end{aligned} \tag{14}$$

Thus (13) follows.

Case 3. For some $t_0 \le \tau_0\epsilon|\ln\epsilon|$, $v(\eta,t_0) = \alpha$. The proof of (13) in this case is very similar to the one in Case 2. This completes the proof of Lemma 1.

Now we use the solution $v(\eta,t)$ of (11) to construct a subsolution of the form

$$\underline{u}(x,t) = v(\phi(x) - Mt, t) - q(t/\epsilon),$$

where $M > 0$ is large, $q(0) = 0$, $q(t/\epsilon)$ is small enough.

We wish to prove that for $0 \le t \le \tau_0 \epsilon |\ln \epsilon|$, $L\underline{u} \le 0$, where

$$\begin{array}{l} L\underline{u} \equiv \epsilon \underline{u}_t - J_\epsilon * \underline{u} + \underline{u} + f(\underline{u}) \\ = \epsilon v_t(\phi(x) - Mt, t) - \epsilon M v_\eta(\phi(x) - Mt, t) - q'(t/\epsilon) - \\ - \int J(y) v(\phi(x - \epsilon y) - Mt, t) dy + v(\phi(x) - Mt, t) + f(v(\phi(x) - Mt, t) - q(t/\epsilon)) \\ = -\epsilon M v_\eta(\phi(x) - Mt, t) - q'(t/\epsilon) + I + f(v(...) - q(t/\epsilon)) - f(v(...)), \end{array} \tag{15}$$

where

$$I = \int J(y) \left[v(\phi(x) - Mt, t) - v(\phi(x - \epsilon y) - Mt, t) \right] dy.$$

We proceed to estimate I as follows. By Taylor's Theorem (we assumed that ϕ has bounded second derivatives),

$$\begin{array}{c} v(\phi(x) - Mt, t) - v(\phi(x - \epsilon y) - Mt, t) = \\ v_\eta(\phi(x) - Mt, t)(\phi(x) - \phi(x - \epsilon y)) + \frac{1}{2} v_{\eta\eta}(\hat{\eta}, t)(\phi(x) - \phi(x - \epsilon y))^2 \\ = v_\eta(\phi(x) - Mt, t)(\epsilon \nabla \phi(x) \cdot y + O(\epsilon^2 |y|^2)) + v_{\eta\eta}(\hat{\eta}, t) O(\epsilon^2 |y|^2). \end{array} \tag{16}$$

This, (13) and the assumption that J be even imply

$$\begin{aligned} I &= v_\eta(\phi(x) - Mt, t) \int_{R^N} J(y) O(\epsilon^2 |y|^2) dy + O(\epsilon) \int_{R^N} J(y) |y|^2 dy \qquad (17) \\ &= v_\eta(\phi(x) - Mt, t) O(\epsilon^2) + O(\epsilon), \quad 0 \le t \le \tau_0 \epsilon |\ln \epsilon|. \qquad (18) \end{aligned}$$

In view of this and (15), it suffices to require

$$\begin{array}{c} 0 \ge -\epsilon(1 + O(\epsilon)) M v_\eta(\phi(x) - Mt, t) - q'(t/\epsilon) + O(\epsilon) \\ + f(v(...) - q) - f(v(...)) \text{ for } 0 \le t \le \tau_0 \epsilon |\ln \epsilon|. \end{array} \tag{19}$$

We assert (to be verified later) that q may be chosen so that it satisfies

$$q(0) = 0, \quad 0 \le q(t/\epsilon) \le 1 - \alpha' \text{ for } 0 \le t \le \tau_0 \epsilon |\ln \epsilon|, \tag{20}$$

where $\alpha' > \alpha$ is close to α.

Observe that there exist small constants $\gamma > 0$, $\mu > 0$, such that if $0 \le q \le 1 - \alpha'$ and $|v - 1| \le \gamma$ or $|v + 1| \le \gamma$, then

$$f(v - q) - f(v) \le -\mu q. \tag{21}$$

Case 1. $1 - \gamma \le v(\phi(x) - Mt, t) \le 1$ or $-1 - \gamma \le v(...) \le -1 + \gamma$, $0 \le t \le \tau_0 \epsilon |\ln \epsilon|$.

We wish (see (19))

$$0 \geq -q'(t/\epsilon) - \mu q(t/\epsilon) + O(\epsilon). \tag{22}$$

Case 2. $-1+\gamma \leq v(\phi(x)-Mt,t) \leq 1-\gamma; \quad 0 \leq t \leq \tau_0\epsilon|\ln\epsilon|$.

We need the following lemma.

Lemma 3. If $-1+\gamma \leq v(\eta,t) \leq 1-\gamma, \quad 0 \leq t$, then there exists a positive constant C_6 such that

$$v_\eta(\eta,t) \geq C_6 > 0. \tag{23}$$

We delay the proof of this until the end of the section. Continuing with Case 2, we note that by (19) and Lemma 3, it suffices to require

$$0 \geq -\frac{\epsilon M}{2}C_6 - q'(t/\epsilon) + bq(t/\epsilon) + O(\epsilon) \tag{24}$$

for $0 \leq t \leq \tau_0\epsilon|\ln\epsilon|$, where $b = \|f'\|_{L^\infty([-2,2])}$.

We take q to satisfy

$$\begin{aligned} &q'(\tau) + \mu q(\tau) = O(\epsilon), \\ &q(0) = 0 \end{aligned} \tag{25}$$

(so that (22) is satisfied). This implies that

$$0 \leq q(\tau) = O(\epsilon)(1-e^{-\mu\tau}) \leq O(\epsilon). \tag{26}$$

Hence (20) is satisfied. Now (24) demands that

$$\frac{\epsilon M C_6}{2} \geq (b+\mu)q(t/\epsilon) = (b+\mu)O(\epsilon).$$

This is true if M is large, independently of small ϵ.

We have thus proved that $\underline{u}(x,t) = v(\phi(x)-Mt,t) - q(t/\epsilon)$ is a subsolution of (7), (8) for $0 \leq t \leq \tau_0\epsilon|\ln\epsilon|$ and hence by the comparison principle discussed below (28),

$$u(x,t) \geq v(\phi(x)-Mt,t) - q(t/\epsilon) \text{ for } 0 \leq t \leq \tau_0\epsilon|\ln\epsilon|. \tag{27}$$

Let $\Omega_0^+ = \{x \in R^N : \phi(x) \geq \alpha + (3+M\tau_0)\sqrt{\epsilon}|\ln\epsilon|\}$. Then if $x \in \Omega_0^+$, $\phi(x) - Mt \geq \alpha + 3\sqrt{\epsilon}|\ln\epsilon|$ for $0 \leq t \leq \tau_0\epsilon|\ln\epsilon|$. Therefore by (26) - (27) and (i) of Lemma 2, we have

$$u(x,\tau_0\epsilon|\ln\epsilon|) \geq 1 - O(\epsilon) \tag{28}$$

This establishes Theorem 1, except for a brief discussion of the comparison principle and the proof of Lemma 3.

Comparison Principle. For simplicity assume $\epsilon = 1$ in (7) and (8). Then the Cauchy problem is equivalent to the following "variation of constants formula"

$$u(\cdot,t) = e^{tA}\phi(\cdot) + \int_0^t e^{(t-s)A} g(u(\cdot,s))ds \equiv F(u),$$

where $Au = J * u$, $g(u) = u + f(u)$. Without loss of generality, assume $g(u)$ is increasing in u (otherwise replace u in (7) by $e^{Mt}u$). Then F is order preserving in the space $X = L^\infty(R^N \times [0,T])$. If $\underline{u}$ is C^1 in the variable t and is a subsolution of (7) and (8), it is routine to show that for small T, F has a unique fixed point $\tilde{u}$ in the unit ball centered at ϕ in the space X; moreover $\tilde{u} \geq \underline{u}$ in $R^N \times [0,T]$. By uniqueness, the solution u of (7), (8) is $\tilde{u}$. A ladder argument then gives $u \geq \underline{u}$ in $R^N \times [0,\infty)$.

Proof of Lemma 3. By easy manipulations, we have

$$v_\eta(\eta,t) = \frac{f_\epsilon(v(\eta,t))}{f_\epsilon(\eta)}$$

(see [9]). Take a small $\beta > 0$ so that $|f'(u)| > C_7$ for some constant $C_7 > 0$ and for $u \in (\alpha-\beta, \alpha+\beta)$ (recall A2).

If $\eta \notin (\alpha-\beta,\alpha+\beta)$, then by the assumption that $-1+\gamma \leq v(\eta,t) \leq 1-\gamma$, we have

$$v_\eta(\eta,t) = \frac{|f(v(\eta,t))|}{|f(\eta)|} \geq \text{ some positive constant } C_8.$$

If $\eta \in (\alpha-\beta,\alpha)$, then by P2 and P3,

$$\begin{aligned} v_\eta(\eta,t) &\geq \tfrac{f(v(\eta,t))}{f_\epsilon(\eta)} \geq \tfrac{f(v(\eta,t))}{\|f'_\epsilon\|_{L^\infty[-2,2]}(\alpha-\eta)} \\ &\geq \begin{cases} \frac{C_7(\alpha-v(\eta,t))}{\|f'_\epsilon\|_\infty(\alpha-\eta)} \text{ if } v(\eta,t) \in (\alpha-\beta,\alpha), \\ \frac{\inf\{f(v): v\in(-1+\gamma,\alpha-\beta)\}}{\|f'_\epsilon\|_\infty \beta} \text{ if } v(\eta,t) \notin (\alpha-\beta,\alpha). \end{cases} \\ &\geq \text{ some positive constant } C_9 \quad (v(\eta,t) < \eta \text{ for } t > 0). \end{aligned}$$

If $\eta \in (\alpha,\alpha+\beta)$ and $v(\eta,t) \geq \alpha+\beta$, then

$$\begin{aligned} v_\eta(\eta,t) = \tfrac{f(v(\eta,t))}{f_\epsilon(\eta)} &\geq \tfrac{\inf\{f(v): v\in(\alpha+\beta,1-\gamma)\}}{\|f'_\epsilon\|_\infty(\eta-\alpha)} \\ &\geq C_{10}, \end{aligned}$$

where C_{10} is some positive constant.

If $\eta \in (\alpha, \alpha+\beta)$ and $v(\eta,t) < \alpha+\beta$, we use the fact that $f'_\epsilon(u) \le 0$ for $u \in (\alpha, \alpha+\beta)$ and small ϵ, to obtain

$$v_\eta(\eta,t) = \frac{-f_\epsilon(v(\eta,t))}{-f_\epsilon(\eta)} \ge \frac{-f_\epsilon(\eta)}{-f_\epsilon(\eta)} = 1.$$

The fact we just used can be proved as follows:

$$\begin{aligned} &f'_\epsilon(u) = (1-\rho_\epsilon(u))f'(u) - \rho'_\epsilon f(u) - \tfrac{\rho_\epsilon}{|\ln\epsilon|} + \rho'_\epsilon\left(\tfrac{\alpha+\sqrt{\epsilon}|\ln\epsilon|-u}{|\ln\epsilon|}\right) \\ &= \rho'_\epsilon(\sqrt{\epsilon} + \tfrac{\alpha-u}{|\ln\epsilon|} - f'(\bar{u})(u-\alpha)) + \text{ plus nonpositive terms.} \end{aligned}$$

But by recalling (A2) that $|f'(u)| > C_7$ on $(\alpha-\beta, \alpha+\beta)$, we see that this last expression is nonpositive for small ϵ.

This establishes the fact and Lemma 3 is proved.

3 Spread of a new phase

The object of this section is to show that if f is such that traveling waves result in the spread of the upper state region at the expense of the lower one, then the upper region also spreads in higher dimensions, provided that initially the solution u of (1) lies above α by some amount in a large enough ball in R^N. The ultimate velocity of this spread is shown to be eventually arbitrarily close to that of 1D traveling waves. Of course the analogous result holds if "upper" and "lower" are interchanged in all of the above. This type of result is known for the equation (3) as well. For example, in 1D it was proved in [11].

Suppose that J is a radial kernel satisfying A1. Define

$$\hat{J}(y_1) = \int_{R^{N-1}} J(y_1, y')dy, \quad y_1 \in R^1, \;\; y' \in R^{N-1}. \tag{29}$$

Then $\hat{J}(y_1)$ is an even function with $\hat{J}(y_1) \ge 0$ and $\int_{R_1} \hat{J}(y_1)dy_1 = 1$.

The one-dimensional problem

$$u_t(y_1,t) = \hat{J} * u - u - f(u) \tag{30}$$

has [4] a traveling wave solution

$$u(y_1,t) = U(y_1 - ct), \quad y_1 \in R^1, \;\; t \ge 0. \tag{31}$$

Suppose $c \neq 0$. Then U is smooth and satisfies

$$
\begin{aligned}
&\hat{J} * U(z) - U(z) + cU'(z) - f(U(z)) = 0, \quad z \in R^1, \\
&U(-\infty) = -1, \quad U(\infty) = 1, \quad U' > 0, \\
&c = \frac{\int_{-1}^{1} f(u)du}{\int_{-\infty}^{\infty} (U')^2 dz}.
\end{aligned} \tag{32}
$$

Thus

$$
\int_{R^N} J(y)U(z - y \cdot n)dy - U(z) + cU'(z) - f(U(z)) = 0, \quad z \in R^1, \tag{33}
$$

for any unit vector n in R^N (which is allowed to depend on z). Consider the Cauchy problem for (1) in R^N, with initial data

$$
u(x,0) = \phi(x), \quad -1 \leq \phi(x) \leq 1, \quad \phi \text{ continuous in } R^N. \tag{34}
$$

Theorem 4. In addition to the above, A1, and A2, suppose the 1D traveling wave U in (31) has velocity $c < 0$. For all small $\eta > 0$, there exists a small $\beta(\eta) > 0$ such that for any $0 < \beta \leq \beta(\eta)$, there exists a positive constant $L(\eta, \beta)$ so that if $\phi(x) \geq \alpha + \eta$ for $|x| \leq L(\eta, \beta)$, then

$$
u(x,t) \geq U(-|x| - ct - \xi(t)) - q(t), \quad x \in R^N, \quad t \geq 0, \tag{35}
$$

where the functions $\xi(t)$ and $q(t)$ depend on η and β and satisfy

$$
\xi'(t)\downarrow \text{ in } t, \quad \xi'(t) = \beta \text{ for } t \text{ large };\tag{36}
$$

$$
\lim_{t \to \infty} q(t) = 0. \tag{37}
$$

($q(t)$ is given by (53) in the proof below.)

Remark. By constructing supersolutions, we can also show that if $\phi(x) \leq \alpha - \eta$ for x near infinity, then an inequality similar to (35) holds, with the inequality sign and the signs of $\xi(t)$ and $q(t)$ changed. These results imply that a large region in which u is close to 1 (the upper state region) propagates at an asymptotic speed $|c|$ in the long run. In one space dimension, we can improve the theorem and the proof is much easier.

Proof of Theorem 4.

We look for a subsolution of the form

$$
\underline{u}(x,t) = U(-|x| - ct - \xi(t)) - q(t),
$$

where $\xi(t)$ and $q(t)$ satisfy the following:

$$\begin{array}{l}\xi'(t) > 0; \\ 0 < q(t) \leq 1 - (\alpha + \eta/2), \quad q(0) = 1 - (\alpha + \eta).\end{array} \tag{38}$$

We need the inequality

$$\begin{array}{l}0 \geq L\underline{u} \equiv \underline{u}_t - J * \underline{u} + \underline{u} + f(\underline{u}) \\ = -(c + \xi'(t))U'(-|x| - ct - \xi(t)) - q'(t) - \\ - \int_{R^N} J(y)U(-|x-y| - ct - \xi(t))dy \\ +U(-|x| - ct - \xi(t)) + f(U(-|x| - ct - \xi(t)) - q(t)).\end{array} \tag{39}$$

By (33), we have

$$\begin{array}{l}\int J(y)U(-|x| - ct - \xi(t) + y \cdot \sigma(x))dy - U(-|x| - ct - \xi(t)) \\ +cU'(-|x| - ct - \xi(t)) - f(U(-|x| - ct - \xi(t))) = 0, \\ \sigma(x) = \frac{x}{|x|} \text{ if } x \neq 0; \text{ any unit vector, if } x = 0.\end{array} \tag{40}$$

We therefore need

$$\begin{array}{l}0 \geq -\xi'(t)U'(-|x| - ct - \xi(t)) - q'(t) + I + \\ +f(U(-|x| - ct - \xi(t)) - q(t)) - f(U(-|x| - ct - \xi(t))),\end{array} \tag{41}$$

where

$$\begin{array}{l}I = \int J(y)\left[U(-|x| - ct - \xi(t) + y \cdot \sigma(x)) - U(-|x-y| - ct - \xi(t))\right] dy \\ = \int_{|y| \geq M(t)} + \int_{|y| \leq M(t)} \equiv I_1 + I_2, \quad M \text{ to be chosen later, } M(t) \geq 1.\end{array} \tag{42}$$

Observe that

$$I_1 \leq \int_{|y| \geq M(t)} 2J(y)dy. \tag{43}$$

To estimate I_2, one sees that if $|x| \geq K(t) \geq M(t)$ (where $K(t)$ will be chosen later), then for $|y| \leq M(t)$, we have by Taylor expansion

$$\begin{array}{l}|x-y| = |x| \left|\sigma(x) - \frac{y}{|x|}\right| = |x| \left[|\sigma(x)| - \sigma(x) \cdot \frac{y}{|x|} + O\left(\frac{|y|^2}{|x|^2}\right)\right] \\ = |x| - \sigma(x) \cdot y + O\left(\frac{M^2(t)}{K(t)}\right).\end{array} \tag{44}$$

Thus if $|x| \geq K(t)$, we have $I_2 \leq \int_{|y| \leq M(t)} J(y)\|U'\|_{L^\infty} O(M^2/K)dy = O(M^2/K)$.

On the other hand if $|x| \leq K(t)$, then

$$\begin{array}{l}I_2 \leq \int_{|y| \leq M(t)} J(y)\left[1 - U(-2K(t) - ct - \xi(t))\right] dy. \\ \leq 1 - U(-2K(t) - ct - \xi(t)) \equiv g(t).\end{array} \tag{45}$$

Summarizing the above estimates, we have

$$I \leq 2 \int_{|y| \geq M(t)} J(y) dy + O\left(\frac{M^2(t)}{K(t)}\right) + g(t) \equiv h(t). \tag{46}$$

Observe that there exist $\mu = \mu(\eta) > 0$, $\gamma = \gamma(\eta) > 0$ such that if $1 - \gamma \leq u \leq 1$ or $-1 \leq u \leq \gamma - 1$ and $0 \leq q \leq 1 - (\alpha + \eta/2)$, then

$$f(u-q) - f(u) \leq -\mu q.$$

Case 1. $1-\gamma \leq U(-|x|-ct-\xi(t)) \leq 1$ or $-1 \leq U(-|x|-ct-\xi(t)) \leq \gamma-1$. We want to have (see (41))

$$q' + \mu q \geq h(t), \quad 0 \leq q \leq 1 - (\alpha + \eta/2), \ q(0) = 1 - (\alpha + \eta) \tag{47}$$

(for the definition of $h(t)$, see (46)).

Case 2. $-1 + \gamma \leq U(-|x| - ct - \xi(t)) \leq 1 - \gamma$. This implies $U'(-|x| - ct - \xi(t)) \geq a > 0$ for some a. In this case we want

$$a\xi'(t) \geq -q'(t) + bq + h(t), \quad b = \|f'\|_{L^\infty([-2,2])} \tag{48}$$

(h given by (46)).

We shall first choose $\xi(t)$, then $q(t)$ so that

$$\begin{aligned} &q' + \mu q = h(t), \\ &q(0) = 1 - (\alpha + \eta), \end{aligned} \tag{49}$$

$$q(t) \leq 1 - (\alpha + \eta/2). \tag{50}$$

To select $\xi(t)$, define $\delta(\eta) = \min\left(\frac{\eta}{4}, \frac{-ca}{2(b+\mu)}, 1 - (\alpha + \eta/2)\right)$. For all $0 < \delta \leq \delta(\eta)$, let $r(t)$ be a smooth nonincreasing function such that

$$r(t) = \begin{cases} \frac{b+\mu}{a}(1 - (\alpha + \eta/2)), & 0 \leq t \leq T, \\ \frac{\delta(b+\mu)}{a}, & 1 + T \leq t < \infty, \end{cases} \tag{51}$$

where T is large enough so that

$$e^{-\mu t} q(0) = e^{-\mu t}(1 - (\alpha + \eta)) < \delta/3 \text{ for } t \geq T. \tag{52}$$

We take $\xi(t)$ such that $\xi'(t) = r(t)$, $t \geq 0$. By (49), we have

$$q(t) = e^{-\mu t}(1 - (\alpha + \eta)) + e^{-\mu t} \int_0^t e^{\mu s} h(s) ds. \tag{53}$$

Now we must verify that q, which satisfies (49), also satisfies (48) and (50). Let us first check (50).

Take $M(t) = (t+m)^{1/4}$, $K(t) = (t+m)^{3/4}$, where m is large enough that

$$\begin{array}{l} K(t) \geq M(t) \text{ for } t \geq 0; \\ h_1(t) \equiv 2\int_{|y|\geq M(t)} J(y)dy + O\left(\frac{M^2}{K}\right) \leq \frac{\delta}{3\mu}, \quad t \geq 0. \end{array} \tag{54}$$

This implies that $q(t) \to 0$ as $t \to \infty$, as desired. It also implies that

$$e^{-\mu t} \int_0^t e^{\mu s} h_1(s) ds \leq \frac{\delta}{3}, \quad t \geq 0. \tag{55}$$

By taking $-\xi(0)$ sufficiently large, we have

$$g(t) = 1 - U(-2K(t) - ct - \xi(t)) \leq \delta/(3\mu) \tag{56}$$

for all $t \geq 0$. (Recall $\xi'(t) = \frac{\delta(b+\mu)}{a} \leq -c/2$ for large $t \geq T+1$).

Now (55) and (56) yield

$$\begin{array}{l} q(t) = e^{-\mu t}(1-(\alpha+\eta)) + e^{-\mu t}\int_0^t e^{\mu s} h_1(s) ds + e^{-\mu t}\int_0^t e^{\mu s} g(s) ds \\ \leq (1-(\alpha+\eta)) + \frac{\delta}{3} + \frac{\delta}{3} \leq 1-(\alpha+\eta) + \frac{\eta}{6} \leq 1-(\alpha+\frac{\eta}{2}). \end{array}$$

Thus (50) holds.

Next, we verify (48), i.e. $a\xi'(t) \geq (b+\mu)q$ for $t \geq 0$. We have

$$a\xi'(t) = ar(t) \geq \begin{cases} (b+\mu)(1-(\alpha+\eta/2)), & 0 \leq t \leq T, \\ (b+\mu)\delta, & t \geq T. \end{cases}$$

For $1 \leq t \leq T$, we have $a\xi'(t) \geq (b+\mu)q(t)$, by (50).

For $t \geq T$, we have

$$\begin{array}{l} a\xi'(t) \geq (b+\mu)\delta \\ \geq (b+\mu)\left[e^{-\mu t}(1-(\alpha+\eta)) + e^{-\mu t}\int_0^t e^{\mu s}(h_1(s)+g(s))ds\right] \\ = (b+\mu)q(t), \end{array}$$

by (52) - (56).

Finally, we have to check that $\phi(x) \geq \underline{u}(x,0) = U(-|x|-\xi(0)) - (1-(\alpha+\eta))$.

For $|x| \leq L$, $\underline{u}(x,0) \leq 1-(1-(\alpha+\eta)) = \alpha+\eta \leq \phi(x)$.

For $|x| \geq L$, $\underline{u}(x,0) \leq U(-L-\xi(0)) - q_0$. Noting that $q_0 = 1-(\alpha+\eta)$, in order to show that $\underline{u}(x,0) \leq -1 \leq \phi(x)$, we need to show that $U(-L-\xi(0)) \leq -(\alpha+\eta)$. In other words, it suffices to show that

$$-L-\xi(0) \leq U^{-1}(-(\alpha+\eta)), \text{ i.e. } -U^{-1}(-(\alpha+\eta)) - \xi(0) \leq L.$$

Now take $\beta(\eta) = \frac{\delta(\eta)(b-\mu)}{a}, \quad \beta = \frac{\delta(b-\mu)}{a}$, with $L(\eta,\beta) = -U^{-1}(-(\alpha+\eta)) - \xi(0)$.

The desired result follows from the comparison principle (discussed below (28)).

4 The propagation of internal layers (phase interfaces)

We return to the analog of the "fast reaction, slow diffusion" case, and denote by $u^\epsilon(x,t)$ the solution of

$$\begin{array}{l} \epsilon u_t^\epsilon = J_\epsilon * u^\epsilon - u^\epsilon - f(u^\epsilon), \qquad x \in R^N, \quad t \geq 0, \\ u^\epsilon(x,0) = \phi(x), \quad -1 \leq \phi(x) \leq 1, \\ J_\epsilon(x) = \frac{1}{\epsilon^N} J\left(\frac{x}{\epsilon}\right), \end{array} \tag{57}$$

with ϕ continuous in R^N.

We look at the problem from a slightly different point of view now, fixing (x,t) and letting $\epsilon\downarrow 0$. We show that the limit is ± 1 except on surfaces analogous to "light cones". This type of result is also known for (5), see e.g. [19].

Before giving the statement of our result, we recall [4] that the 1D problem (30) always has a monotone traveling wave U (see (31)). If $c \neq 0$, then U is C^1 smooth and satisfies (32). If $c = 0$, U may be not differentiable and may even have jump discontinuities at some points. In this section, we assume that the set of such points is finite, which is true if $f(u)$ has only finitely many critical points over interval $[-1,1]$. At points where U is differentiable, U is C^1 and $U' > 0$. In any event, $c = 0$ or $c \neq 0$, U satisfies (33), possibly except at finitely many points.

Theorem 5. Assume A1, A2 and that J is radial. Let $\Omega_0^+ = \{x \in R^N \big| \phi(x) > \alpha\}$. Then the following is true:

(i) If the one-dimensional traveling wave (31) has velocity $c < 0$, then for any $t > 0$,

$$\lim_{\epsilon \to 0^+} u^\epsilon(x,t) = \begin{cases} 1, \text{ if } \mathrm{dist}(x,\Omega_0^+) < -ct, \\ -1, \text{ if } \mathrm{dist}(x,\Omega_0^+) > -ct, \end{cases} \tag{58}$$

A similar assertion holds if $c > 0$.

(ii) If the one-dimensional traveling wave has velocity $c = 0$, then for any $t > 0$,

$$\lim_{\epsilon\to 0^+} u^\epsilon(x,t) = \begin{cases} 1, \text{ if } \phi(x) > \alpha, \\ -1, \text{ if } \phi(x) < \alpha. \end{cases} \tag{59}$$

Proof of Thm. 5(i).

First, let (x_0, t_0) be any point such that $\mathrm{dist}(x_0, \Omega_0^+) < -ct_0$. There is a point $x_1 \in \Omega_0^+$ such that $|x_0 - x_1| < -ct_0$. Without loss of generality, we assume $x_1 = 0$. Take a small ball $B_{r_0}(0)$ so that $\overline{B_{r_0}(0)} \subset \Omega_0^+$. Then there exists a positive constant $\eta > 0$ such that $\phi(x) \geq \alpha + \eta$ for $|x| \leq r_0$. Let $\beta(\eta)$ be the small positive constant in the statement of Theorem 4. Take a small number $\beta \in (0, \beta(\eta))$ so that

$$|x_0| = |x_0 - x_1| < -ct_0 - 2\beta t_0. \tag{60}$$

Now rescale $u^\epsilon(x,t)$ as follows: $v^\epsilon(x,t) = u^\epsilon(\epsilon x, \epsilon t)$. Then v^ϵ satisfies

$$v_t^\epsilon(x,t) = J * v^\epsilon - v^\epsilon - f(v^\epsilon), \quad x \in R^N, \quad t \geq 0, \tag{61}$$

$$v^\epsilon(x,0) = \phi(\epsilon x) \equiv \phi_\epsilon(x), \qquad -1 \leq \phi_\epsilon \leq 1. \tag{62}$$

Note that $\phi_\epsilon(x) \geq \alpha + \eta$ for $|x| \leq r_0/\epsilon$.

Now by Theorem 4, there exist functions $\xi(t)$ and $q(t)$ depending only on η and β such that for all small $\epsilon > 0$,

$$\begin{aligned} &v^\epsilon(x,t) \geq U(-|x| - ct - \xi(t)) - q(t), \quad x \in R^N, \quad t \geq 0, \\ &\xi'(t) = \beta \text{ for } t \geq T(\eta,\beta), \text{ and } q(t)\to 0 \text{ as } t\to\infty. \end{aligned} \tag{63}$$

We therefore have that

$$u^\epsilon(x_0,t_0) \geq U\left(\frac{-|x_0| - ct_0}{\epsilon} - \xi(t_0/\epsilon)\right) - q(t_0/\epsilon). \tag{64}$$

But since $\xi'(t) = \beta$ for $t \geq T(\eta,\beta)$, we have $\xi(t_0/\epsilon) \leq \frac{3\beta t_0}{2\epsilon}$ for ϵ small. Hence by (60)

$$\begin{aligned} u^\epsilon(x_0,t_0) &\geq U\left(\tfrac{-|x_0|-ct_0}{\epsilon} - \tfrac{3\beta t_0}{2\epsilon}\right) - q(t_0/\epsilon) \\ &\geq U\left(\tfrac{\beta t_0}{2\epsilon}\right) - q(t_0/\epsilon) \to 1 \text{ as } \epsilon\to 0^+. \end{aligned} \tag{65}$$

Thus $\lim_{\epsilon\to 0^+} u^\epsilon(x_0,t_0) = 1$.

It remains to show the second part of (i), i.e.

$$\lim_{\epsilon\to 0^+} u^\epsilon(x,t) = -1 \text{ for } \mathrm{dist}(x,\Omega_0^+) > -ct, \quad t > 0. \tag{66}$$

Let (x_0, t_0) be such that $\mathrm{dist}(x_0, \Omega_0^+) > -ct_0$ (we take $x_0 = 0$). There are a small $\delta > 0$ and constants $\xi_1 > \xi_0$ such that $\xi_1 > \xi_0 > -c_\delta t_0$, where $c_\delta = c - \delta < 0$ and the closure of the ball $B_{\xi_1}(0)$ does not intersect $\overline{\Omega_0^+}$. Then there exists a small $\eta > 0$ such that $\phi(x) \leq \alpha - \eta$ for $|x| \leq \xi_1$. To show (66), it suffices to show that for all small $\beta > 0$,

$$\limsup_{\epsilon \to 0^+} u^\epsilon(x,t) \leq -1 + \beta. \tag{67}$$

To this end, we construct a supersolution of the form

$$\bar{u}(x,t) = \ U\left(\frac{|x| - c_\delta t - \xi_0}{\epsilon} + r(t/\epsilon)\right) + q(t/\epsilon). \tag{68}$$

We shall choose $r(t)$ and $q(t)$ so that they satisfy

$$\begin{array}{l} 0 < q(t) \leq 1 + \alpha - \frac{\eta}{2}, \quad q(0) = 1 + (\alpha - \eta), \\ r'(t) \geq 0, \quad r'(t) \equiv 0 \text{ for large } t. \end{array} \tag{69}$$

Then

$$\begin{array}{l} L\bar{u}(x,t) = \epsilon \bar{u}_t(x,t) - J_\epsilon * \bar{u} + \bar{u} + f(\bar{u}) \\ = (-c_\delta + r'(t/\epsilon))U'\left(\frac{|x| - c_\delta t - \xi_0}{\epsilon} + r(t/\epsilon)\right) + q'(t/\epsilon) + \\ + U\left(\frac{|x| - c_\delta - \xi_0}{\epsilon} + r(t/\epsilon)\right) + f(U(...) + q(t/\epsilon)) - \\ - \int_{R^N} J_\epsilon(y) U\left(\frac{|x-y| - c_\delta t - \xi_0}{\epsilon} + r(t/\epsilon)\right) dy. \end{array} \tag{70}$$

Setting the vector n in (33) equal to σ in (40), we have

$$\begin{array}{c} \int_{R^N} J(y) U\left(\frac{|x| - c_\delta t - \xi_0}{\epsilon} + r(t/\epsilon) - \sigma(x) \cdot y\right) dy - U\left(\frac{|x| - c_\delta t - \xi_0}{\epsilon} + r(t/\epsilon)\right) \\ + cU'(...) - f(U(...)) = 0. \end{array} \tag{71}$$

Thus

$$\begin{array}{l} L\bar{u}(x,t) = (\delta + r'(t/\epsilon))U'\left(\frac{|x| - c_\delta t - \xi_0}{\epsilon} + r(t/\epsilon)\right) + q'(t/\epsilon) \\ + f(U(...) + q(t/\epsilon)) - f(U(...)) + I, \end{array} \tag{72}$$

where, for some M_β to be chosen,

$$\begin{array}{l} I = \int_{R^N} J(y) \left[U\left(\frac{|x| - c_\delta t - \xi_0}{\epsilon} + r(t/\epsilon) - \sigma(x) \cdot y\right) - U\left(\frac{|x - \epsilon y| - c_\delta t - \xi_0}{\epsilon} + r(t/\epsilon)\right)\right] \\ = \int_{|y| \geq M_\beta} + \int_{|y| \leq M_\beta} \equiv I_1 + I_2. \end{array} \tag{73}$$

We need the inequality $L\bar{u} \geq 0$ for $0 \leq t \leq t_0$, ϵ small. For all small $\beta > 0$, there is a large number $M_\beta > 0$ such that

$$I_1 \geq -2 \int_{|y| \geq M_\beta} J(y) dy > -\beta/100. \tag{74}$$

For $|x| \geq \frac{1}{2}(\xi_0 + c_\delta t_0) > 0$ and ϵ small, by Taylor's theorem as in the previous section, we have

$$|I_2| \leq \int_{|y| \leq M_\beta} J(y) O(\epsilon) dy < \beta/100 \text{ for all } t \geq 0. \tag{75}$$

For $|x| \leq \frac{1}{2}(\xi_0 + c_\delta t_0)$, small $\epsilon > 0$ and $0 \leq t \leq t_0$,

$$\begin{aligned} I_2 &\geq \textstyle\int_{|y| \leq M_\beta} J(y) \left[-1 - U\left(\frac{|x-\epsilon y| - c_\delta t - \xi_0}{\epsilon} + r(t/\epsilon)\right)\right] dy \\ &\geq \textstyle\int_{|y| \leq M_\beta} J(y) \left[-1 - U\left(\frac{-c_\delta t - \xi_0}{3\epsilon} + r(t/\epsilon)\right)\right] dy \\ &\geq -1 - U\left(\frac{-c_\delta t_0 - \xi_0}{3\epsilon} + r(t/\epsilon)\right) \equiv g(t). \end{aligned} \tag{76}$$

Now let $\mu = \mu(\eta) > 0$ and $\gamma = \gamma(\eta) > 0$ be constants such that if $0 \leq q \leq 1 + (\alpha - \frac{\eta}{2})$ and $1 - \gamma \leq u \leq 1$ or $-1 \leq u \leq \gamma - 1$, then

$$f(u+q) - f(u) \geq \mu q.$$

We consider several cases.

Case 1. $1 - \gamma \leq U\left(\frac{|x| - c_\delta t - \xi_0}{\epsilon} + r(t/\epsilon)\right) \leq 1$ or $-1 \leq U(...) \leq \gamma - 1$, and $|x| \geq \frac{1}{2}(\xi_0 + c_\delta t_0) > 0$, for $0 \leq t \leq t_0$.

In this case, we want (see (69), (72) - (75))

$$0 \leq q'(t/\epsilon) + \mu q(t/\epsilon) - \beta/50. \tag{77}$$

Case 2. $1 - \gamma \leq U\left(\frac{|x| - c_\delta t - \xi_0}{\epsilon} + r(t/\epsilon)\right) \leq 1$ or $-1 < U(...) \leq \gamma - 1$, and $|x| \leq \frac{1}{2}(\xi_0 + c_\delta t_0) > 0$, for $0 \leq t \leq t_0$.

In this case, we want (see (72) - (74) and (76))

$$0 \leq q'(t/\epsilon) + \mu q(t/\epsilon) - \beta/100 + g(t) \tag{78}$$

(here $g(t)$ is given in (76)).

Case 3. $-1+\gamma \leq U\left(\frac{|x| - c_\delta t - \xi_0}{\epsilon} + r(t/\epsilon)\right) \leq 1-\gamma$ and $|x| \geq \frac{1}{2}(\xi_0 + c_\delta t_0) > 0$, for $0 \leq t \leq t_0$.

Let a be such that $U'(...) \geq a > 0$ and $b = \|f'\|_{L^\infty([-2,2])}$. In this case, we want (see (72) - (75))

$$ar'(t/\epsilon) + a\delta + q'(t/\epsilon) - bq(t/\epsilon) - \beta/50 \geq 0. \tag{79}$$

Case 4. $-1+\gamma \leq U\left(\frac{|x|-c_\delta t-\xi_0}{\epsilon} + r(t/\epsilon)\right) \leq 1-\gamma$ and $|x| \leq \frac{1}{2}(\xi_0 + c_\delta t_0)$ for $0 \leq t \leq t_0$.

In this case, we want (see (72) - (74) and (76))

$$ar'(t/\epsilon) + a\delta + q'(t/\epsilon) - bq(t/\epsilon) - \beta/100 + g(t) \geq 0. \tag{80}$$

We take q to satisfy

$$q'(\tau) + \mu q(\tau) - \beta = 0, \quad q(0) = 1 + \alpha - \eta,$$

so that

$$q(\tau) = (1+\alpha-\eta)e^{-\mu\tau} + \frac{\beta}{\mu}(1 - e^{-\mu\tau}).$$

Take $\beta > 0$ so small that

$$\beta < \frac{\delta a}{2} \text{ and } \beta < \frac{\mu\delta a}{4(b+\mu)}. \tag{81}$$

Let τ_0 be such that

$$-\frac{a\delta}{2} + (1+\alpha-\eta)(b+\mu)e^{-\mu\tau_0} + \frac{\beta(b+\mu)}{\mu} = 0.$$

Such τ_0 exists by (81) and by assuming δ is small enough (no loss of generality). We take $r(\tau)$ given by

$$ar'(\tau) = \begin{cases} \frac{-a\delta}{2} + (b+\mu)\left[(1+\alpha-\eta)e^{-\mu\tau} + \beta/\mu\right], & \text{if } 1 \leq \tau \leq \tau_0; \\ 0, & \text{if } \tau \geq \tau_0. \end{cases}$$
$$r(0) = 0.$$

Thus $r' \geq 0$ and $r'(\tau) \equiv 0$ for $\tau \geq \tau_0$. Now we check whether (69) and (77) - (80) are satisfied.

The inequality (69) will be satisfied if β is taken small enough, which involves no loss of generality. Also, (77) is obviously true.

To show (78), we just need to show that

$$\beta/100 \geq -g(t) = 1 + U\left(\frac{-c_\delta t_0 - \xi_0}{3\epsilon} + r(t/\epsilon)\right).$$

This is true for small ϵ, because

$$-g(t) \leq 1 + U\left(\frac{-c_\delta t_0 - \xi_0}{3\epsilon} + r(\tau_0)\right) \to 0 \text{ as } \epsilon \downarrow 0.$$

To prove (79) and (80), we merely need to show that

$$-a\delta + (b+\mu)q(t/\epsilon) + 99\beta/100 - g(t) \leq ar'(t/\epsilon). \tag{82}$$

Since $-g(t) \leq \beta/100$ and in view of (81), to show (82) it suffices to establish that

$$(b+\mu)q(\tau) - a\delta/2 \leq ar'(\tau),$$

which will follow if

$$(b+\mu)\left[(1+\alpha-\eta)e^{-\mu\tau} + \frac{\beta}{\mu}\right] - a\delta/2 \leq ar'(\tau).$$

But this is true by the definitions of $r(\tau)$ and τ_0.

Now to show that $\bar{u}$ is a supersolution of (57) for $0 \leq t \leq t_0$, we only need to demonstrate that

$$\bar{u}(x,0) \geq \phi(x)$$

in R^N, which by (68) is true if and only if

$$U\left(\frac{|x| - \xi_0}{\epsilon}\right) + (1+\alpha-\eta) \geq \phi(x) \tag{83}$$

in R^N. But since $\phi(x) \leq \alpha - \eta$ for $|x| \leq \xi_1$, (83) holds in the ball $B_{\xi_1}(0)$.

For $|x| \geq \xi_1 > \xi_0$,

$$U\left(\frac{|x| - \xi_0}{\epsilon}\right) + (1+\alpha-\eta) \geq U\left(\frac{\xi_1 - \xi_0}{\epsilon}\right) + (1+\alpha-\eta) \geq 1 \geq \phi(x)$$

if ϵ is small. Thus (83) holds in R^N and $\bar{u}(x,t)$ is a supersolution of (57) for $0 \leq t \leq t_0$.

By the comparison principle below (28), we have

$$u^\epsilon(x,t) \leq U\left(\frac{|x| - c_\delta t - \xi_0}{\epsilon} + r(\tau_0)\right) + q(t/\epsilon), \quad 0 \leq t \leq t_0.$$

In particular (note $0 = x_0$),

$$u^\epsilon(x_0,t_0) \leq U\left(\frac{-c_\delta t_0 - \xi_0}{\epsilon} + r(\tau_0)\right) + (1+\alpha-\eta)e^{-\mu t_0/\epsilon} + \frac{\beta}{\mu}(1 - e^{-\mu t_0/\epsilon}).$$

Sending $\epsilon\downarrow 0$ and using the fact that $c_\delta t_0 + \xi_0 > 0$, we obtain

$$\limsup_{\epsilon\downarrow 0} u^\epsilon(0,t_0) \leq -1 + \frac{\beta}{\mu}.$$

This proves the second part of (i).

Proof of Thm. 5(ii). For every x_0 with $\phi(x_0) < \alpha$ and every $t_0 > 0$, we need to prove that $\lim_{\epsilon\downarrow 0} u^\epsilon(x_0,t_0) = -1$. Without loss of generality, take $x_0 = 0$. Clearly for some small $\xi_1 > 0$, the ball $B_{\xi_1}(0)$ does not intersect $\overline{\Omega_0^+}$. Suppose $\phi(x) \leq \alpha - \eta$ in this ball, for some $\eta > 0$. Take small positive constants δ and ξ_0 such that $\delta t_0 < \xi_0 < \xi_1$. As in the proof of the second part of (i), we can show that

$$\bar{u}(x,t) = U\left(\frac{|x| + \delta t - \xi_0}{\epsilon} + r(t/\epsilon)\right) + q(t/\epsilon)$$

is a supersolution for $0 \leq t \leq t_0$ and small ϵ, where

$$q(\tau) = (1 + \alpha - \eta)e^{-\mu\tau} + \frac{\beta}{\mu}(1 - e^{-\mu\tau}),$$

$$r'(\tau) \geq 0, \quad r'(\tau) \equiv 0 \text{ for large } \tau, \quad r(0) = 0.$$

This, as in the previous proof, yields

$$\lim_{\epsilon\downarrow 0} u^\epsilon(0,t_0) = -1.$$

(When U is not C^1 at some points (only finitely many, by our assumption), after showing that $L\bar{u} > 0$ and $\bar{u}(x,0) > \phi(x)$, we use the comparison arguments in the proof of Theorem 4.1 in [4] to get $\bar{u} \geq u^\epsilon$.)

Similarly, we can prove the remaining part of (ii). This completes the proof of Theorem 5.

5 Motion by curvature

As in the case of the better known equation (5) and also in the case considered in [21], when the velocity c of the 1D traveling wave vanishes, then the motion of interfaces in higher dimensions is slower, by a factor ϵ, and is driven by curvature of the interface. In this section we examine this case, rescaling time in accordance with this expectation. Denoting the new time variable by the same symbol t, we consider the equation

$$\epsilon^2 u_t = J_\epsilon * u - u - f(u). \tag{84}$$

We consider families $u(x,t;\epsilon)$ of solutions with the layer property, where $x \in R^2, \quad t \in R^+$, defined for all small enough positive ϵ. In addition to A1, we assume, for some constant C and all x,

$$J(x)|x|^4 \leq C. \tag{85}$$

And in addition to A2,

$$f'(u) > -1, \qquad \int_{-1}^{1} f(u)du = 0. \tag{86}$$

Our object is to investigate formally the properties of such layered solution families, in particular the manner in which they move.

Let $\Gamma(t;\epsilon)$ denote a time-dependent closed curve in the plane, assumed to be the locus where $u(x,t;\epsilon) = 0$ for a solution u of (84). We suppose that u has a "layer" at Γ in the sense implied by the matched asymptotic construction below, and deduce the local structure of u near there, as well as the law by which Γ moves.

At locations bounded away from Γ, we suppose that u can be described by an "outer" expansion

$$u(x,t;\epsilon) = u_0(x,t) + \epsilon u_1(x,t) + ... \tag{87}$$

Using the variable $\xi' = (x - x')/\epsilon$, we may express the convolution in the form

$$J_\epsilon * w(x) = \int_{R^2} J(\xi')w(x + \epsilon\xi')d\xi'. \tag{88}$$

Also note that

$$u(x + \epsilon\xi', t; \epsilon) = u_0(x,t) + \epsilon[u_1(x,t) + \xi' \cdot \nabla u_0(x,t)] + ... \tag{89}$$

Substituting these expressions into (84), we obtain

$$f(u_0) + \epsilon[f'(u_0)u_1 - (\int_{R^2} J(\xi')\xi' d\xi') \cdot \nabla u_0(x,t)] + O(\epsilon^2) = 0. \tag{90}$$

Therefore $f(u_0) = 0$ and the coefficient of ϵ equals 0. For u_0 we select the outermost zeros of f:

$$u_0(x,t) = \pm 1; \tag{91}$$

now u_1, etc., can in turn be determined successively from (90) and its extensions. The choice $u_0 = 1$ will be used on one side of Γ (we refer to this region as $\mathcal{D}_+$), and $u_0 = -1$ in $\mathcal{D}_-$.

To construct an inner expansion, we shall establish a local orthogonal coordinate system $(r(x,t;\epsilon), s(x,t;\epsilon))$ as in [5, 12, 18] in which $r = \pm\,\mathrm{dist}(x,\Gamma(t;\epsilon))$ with $r>0$ in $\mathcal{D}_+$. When $r=0$, s measures arclength along Γ from some point on Γ which moves (when Γ moves) in a trajectory orthogonal to Γ.

In some neighborhood $\mathcal{N}$ of $\Gamma(t;\epsilon)$, we may invert to obtain $x(r,s,t;\epsilon)$. By possibly shrinking $\mathcal{N}$, we know that there is a number $R>0$ such that the coordinate transformation is in fact well defined and invertible in any ball of radius R centered at any point in $\mathcal{N}$. Let $B(x)$ denote the ball of that radius centered at x. By (85), we have that for $x' \notin B(x)$, $J_\epsilon(x-x') \le C|x-x'|^{-4}\epsilon^2$, so that if the quantity $(J_\epsilon * u)(x)$ is replaced by the integral only over $B(x)$, with the same integrand, we will incur an error only of the order $O(\epsilon^2)$. In the following inner expansion, we shall disregard terms of that order in ϵ, so will replace convolutions by integrals over $B(x)$.

This means that we may transform to the new (moving) coordinate system (r,s) in the convolution integral (suppressing dependence of r, s, u on t and ϵ) as follows:

$$J_\epsilon * u = \int_{B(x)} J_\epsilon(x(r,s)-x(r',s'))u(r',s')\left|\frac{\partial x'}{\partial(r',s')}\right| dr'ds' + O(\epsilon^2). \tag{92}$$

Note that

$$\frac{\partial x}{\partial(r,s)} = 1 + r\kappa \tag{93}$$

(see (95) below).

We expand

$$\begin{aligned} x(r',s') - x(r,s) &= (r'-r)x_r(r,s) + (s'-s)x_s(r,s) + \tfrac{1}{2}(r'-r)^2 x_{rr} + \\ &(r'-r)(s'-s)x_{rs} + \tfrac{1}{2}(s'-s)^2 x_{ss} + O(d^3), \end{aligned} \tag{94}$$

where $d^2 = (r'-r)^2 + (s'-s)^2$.

Let $n(s)$, $\tau(s)$ be the unit normal and tangential vectors to Γ, oriented so that n points into $\mathcal{D}^+$ and τ points in the direction of increasing s. Let $\kappa(s)$ be the curvature, so that

$$\begin{aligned} &n_s(s) = \kappa\tau, \quad \tau_s = -\kappa n, \quad x_r(r,s) = n(s), \quad x_s(r,s) = \tau(s)(1+r\kappa), \\ &x_{rr}(r,s) = n_r = 0, \quad x_{rs} = n_s = \kappa\tau, \quad x_{ss} = -\kappa n(1+r\kappa) + r\tau(s)\kappa'(s). \end{aligned} \tag{95}$$

On the basis of this and (94),

$$\begin{aligned}&x(r',s')-x(r,s)=\\&n(s)(r'-r)+\tau(s)(1+r\kappa)(s'-s)+\\&+\kappa\tau(r'-r)(s'-s)-\tfrac{1}{2}(\kappa(1+r\kappa)n-r\kappa'(s)\tau)(s'-s)^2+O(d^3).\end{aligned} \tag{96}$$

Let s_0 be a fixed point on Γ. We define

$$\rho=\frac{r}{\epsilon},\quad \sigma=\frac{s-s_0}{\epsilon}.$$

Thus from (96), we have

$$\frac{x(r',s')-x(r,s)}{\epsilon}=n(\rho'-\rho)+\tau(\sigma'-\sigma)+\epsilon\kappa[\tau\rho'(\sigma'-\sigma)-\frac{1}{2}n(\sigma'-\sigma)^2]+O(\epsilon^2). \tag{97}$$

We assume that u can be expressed in the form

$$u(r,s,t;\epsilon)=U(\rho,r,s,t;\epsilon), \tag{98}$$

with U a regular function of the variables shown. The form (98) itself expresses a certain type of dependence of u on ϵ.

In light of (97), we obtain

$$\begin{aligned}&J_\epsilon(x(r,s)-x(r',s'))=\tfrac{1}{\epsilon^2}J(\tfrac{x(r',s')-x(r,s)}{\epsilon})\\&=\tfrac{1}{\epsilon^2}J(n(\rho'-\rho)+\tau(\sigma'-\sigma)+\epsilon\kappa[\tau\rho'(\sigma'-\sigma)-\tfrac{1}{2}n(\sigma'-\sigma)^2]+O(\epsilon^2))\\&=\tfrac{1}{\epsilon^2}\left[J((\rho'-\rho)n+(\sigma'-\sigma)\tau)+\epsilon\kappa\left[\partial_\tau J(...)\rho'(\sigma'-\sigma)-\tfrac{1}{2}\partial_n J(...)(\sigma'-\sigma)^2\right]+O(\epsilon^2)\right].\end{aligned} \tag{99}$$

And from this, (92), and (93), we find (suppressing t-dependence for the moment)

$$\begin{aligned}&J_\epsilon * U=\\&\textstyle\int_{B(x)}[J((\rho'-\rho)n+(\sigma'-\sigma)\tau)+...]\,U(\rho',\epsilon\rho',\epsilon\sigma'+s_0)(1+\epsilon\rho'\kappa)d\rho'd\sigma'+O(\epsilon^2)\\&=L_0U+\epsilon L_1U+O(\epsilon^2),\end{aligned} \tag{100}$$

where

$$L_0U=\int_{B(x)}J((\rho'-\rho)n+(\sigma'-\sigma)\tau)U(\rho',0,s_0)d\rho'd\sigma', \tag{101}$$

$$\begin{aligned}&L_1U=\textstyle\int_{B(x)}\{J((\rho'-\rho)n+(\sigma'-\sigma)\tau)(\rho'U_r(\rho',0,s_0)+\sigma'U_s(...))+\\&\kappa(\partial_\tau J(...)\rho'(\sigma'-\sigma)-\tfrac{1}{2}\partial_n J(...)(\sigma'-\sigma)^2+\rho'J(...))U(\rho',0,s_0)\}d\rho'd\sigma'\\&\equiv L_{11}U_r+L_{12}U_s+\kappa L_{13}U.\end{aligned} \tag{102}$$

In the inner expansion, we construct approximate solutions $U(\rho, r, s)$ which are defined and bounded for all real values of ρ. Again because of (85), the range of the integration variable ρ' in (101) may be extended to $-\infty < \rho' < \infty$, incurring only an additional error of the order $O(\epsilon^2)$, which can be ignored. The same is true of the integration variable σ'.

Except for this error, we may therefore express $L_0 U$ in the form

$$L_0 U = \int_{-\infty}^{\infty} \hat{J}(s, \rho' - \rho) U(\rho', 0, s_0) d\rho', \tag{103}$$

where

$$\hat{J}(s, \rho' - \rho) = \int_{-\infty}^{\infty} J((\rho' - \rho) n(s) + \sigma' \tau(s)) d\sigma'. \tag{104}$$

In this last integral, s is fixed.

We now construct an inner expansion of the equation (84) near Γ. We recall that (r, s) is a function of t. In fact,

$$\frac{\partial U}{\partial t} = \frac{1}{\epsilon} U_\rho r_t + O(1). \tag{105}$$

Since the normal velocity v of Γ in the direction of $\mathcal{D}_+$ is equal to $-r_t$, we have $\epsilon^2 U_t = -\epsilon v U_\rho + O(\epsilon^2)$, and the expansion becomes the follows:

$O(1)$:

$$0 = L_0 U_0 - U_0 - f(U_0), \tag{106}$$

$O(\epsilon)$:

$$-v U_{0\rho} = L_1 U_0 + L_0 U_1 - U_1 - f'(U_0) U_1. \tag{107}$$

By (86) and [4], the equation (106) has a unique solution taking the value α at $\rho = 0$:

$$U_0 = \psi(\rho, s). \tag{108}$$

It is a smooth stationary profile of the 1D nonlinear convolution equation. It depends on s through the kernel $\hat{J}$ of the operator L_0 (103). Set

$$\mathcal{L} = L_0 - 1 - f'(\psi(\rho, s)).$$

We then obtain from (107)

$$\mathcal{L} U_1 = -v U_{0\rho} - L_1 U_0 = -v \psi_\rho - L_1 \psi(\rho, s). \tag{109}$$

We multiply this by $\psi_\rho(\rho, s)$ and integrate with respect to ρ. Use the fact that ψ is independent of r (while depending on ρ) and that ψ_ρ is the kernel of $\mathcal{L}$. We obtain (using the L_2-scalar product)

$$\langle -v\psi_\rho - L_1 \psi, \psi_\rho \rangle = 0, \tag{110}$$

hence

$$v\|\psi_\rho\|^2 = -\langle L_{12}\psi_s, \psi_\rho\rangle - \kappa\langle L_{13}\psi, \psi_\rho\rangle. \tag{111}$$

To investigate the derivative $\psi_s(\rho, s)$, we differentiate (106), with the substitution (108), with respect to s. We thus obtain

$$\mathcal{L}\psi_s = -\int \hat{J}_s(s, \rho' - \rho)\psi(\rho', s)d\rho'. \tag{112}$$

Using (104) we find an expression for $\hat{J}_s$:

$$\begin{array}{l} \hat{J}_s(s, \rho' - \rho) = \int_{-\infty}^{\infty} \nabla J \cdot [(\rho' - \rho)n_s + \sigma'\tau_s]d\sigma' \\ = \kappa \int_{-\infty}^{\infty} \nabla J \cdot [(\rho' - \rho)\tau(s) - \sigma' n(s)]d\sigma' \\ = \kappa \int_{-\infty}^{\infty} (-\partial_n J(...)\sigma' + \partial_\tau J(...)(\rho' - \rho))\, d\sigma' \\ \equiv \kappa M(s, \rho' - \rho). \end{array} \tag{113}$$

Thus

$$\psi_s(\rho, s) = \kappa(\mathcal{L})^{-1} \int M(s, \rho' - \rho)\psi(\rho', s)d\rho', \tag{114}$$

where $(\mathcal{L})^{-1}$ denotes a suitable inverse (obtained by noting that $\psi_s(0, s) = 0$). In this connection we note that by (112), it is being applied to a function which is guaranteed to be in the range of $\mathcal{L}$.

Now combining (114) and (111), we obtain

$$v\|\psi_\rho\|^2 = -\kappa\langle M_1\psi, \psi_\rho\rangle, \tag{115}$$

for an operator M_1 which can be determined from the above.

This is the motion by curvature law for these interfaces. It says that the normal velocity of Γ is proportional to κ, with the constant of proportionality dependent in a specific way on J and ψ. It also depends on the orientation of the interface through the dependence on s. In fact, both the norm on the left of (115) and the scalar product on the right depend on s.

When J is isotropic, i.e., radial, $\psi = \psi(\rho)$ is independent on s. Then by (111), we can compute the velocity v explicitly as follows.

$$v\|\psi_\rho\|^2 = -\kappa\langle L_{13}\psi, \psi_\rho\rangle, \tag{116}$$

where

$$\begin{array}{l} \langle L_{13}\psi, \psi_\rho\rangle = \int \psi'(\rho)d\rho \int\int \{\partial_\tau J((\rho' - \rho)n + (\sigma' - \sigma)\tau)\rho'(\sigma' - \sigma) - \\ \frac{1}{2}\partial_n J((\rho' - \rho)n + (\sigma' - \sigma)\tau)(\sigma' - \sigma)^2 + \rho' J(...)\}\psi(\rho')d\sigma' d\rho' \\ = \int \psi'(\rho)d\rho \int\int \{\partial_{\sigma''} J((\rho' - \rho)n + \sigma''\tau)\rho'\sigma'' - \frac{1}{2}\partial_n J((\rho' - \rho)n + \sigma''\tau)(\sigma'')^2 + \\ \rho' J((\rho' - \rho)n + \sigma''\tau)\}\psi(\rho')d\sigma'' d\rho'. \end{array} \tag{117}$$

Integrating by parts on the first integrand with respect to the σ'' variable, we see that it cancels with the third integrand. Thus

$$\begin{aligned}
&\langle L_{13}\psi, \psi_\rho\rangle = -\tfrac{1}{2}\int \psi'(\rho)d\rho \int\int \partial_n J((\rho'-\rho)n + \sigma''\tau)(\sigma'')^2\psi(\rho')d\sigma'' d\rho'. \\
&= \tfrac{1}{2}\int\int\{\psi'(\rho)(\sigma'')^2[\tfrac{d}{d\rho}\int J((\rho-\rho')n+\sigma''\tau)\psi(\rho')d\rho']\}d\rho d\sigma'' \\
&= \tfrac{1}{2}\int\int\{\psi'(\rho)(\sigma'')^2[\int J(\rho' n+\sigma''\tau)\psi'(\rho-\rho')d\rho']\}d\rho d\sigma''
\end{aligned} \tag{118}$$

by isotropy, where the argument (ρ', σ'') of J is the point with those coordinates with respect to the standard orthonormal pair of coordinate axes.

It is important to note that while the scalar product is positive in the isotropic case (because ψ is increasing), it may well be negative for some values of the orientation when J is anisotropic. For those values, the motion by curvature law (115) is ill-posed and it is unreasonable to suppose that there exist regular layered solutions with such orientations. Excluded angles are also a common phenomenon in the theory of crystal growth.

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