Contents

1 The Main Problem **1**

2 Solution Approach **3**

2.1 1D Interpolation . . . 4

2.2 1D Boundary Value Problems . . . 4

2.3 2D single unknown test problem . . . 6

2.4 2D, 2 unknown tests . . . 7

2.5 2D, 2 unknown, 2 domain tests . . . 8

2.5.1 Test problems with bounded derivatives . . . 8

2.5.2 Test problems with unbounded derivatives . . . 9

2.6 Current Work . . . 11

1 The Main Problem

For this problem, the isotropic, static Navier's equations

$$\nabla^2\mathbf{u} + \frac{m}{m-2}\nabla(\nabla\cdot\mathbf{u}) = 0 \tag{1}$$

need to be solved on the two domains shown in figure 1, subject to suitable boundary conditions involving displacements and stresses (values and derivatives of the unknowns). Here, $\mathbf{u}$ is the displacement vector, and m depends on material constants. Since the problem has circular symmetry, a polar coordinate system will be used. In this system, the equations become

$$\begin{aligned}
&\frac{m\left(\frac{\partial^2}{\partial z\,\partial r}u_3(r,\theta,z)\right)}{m-2} + \frac{\frac{\partial^2}{\partial\theta^2}u_1(r,\theta,z)}{r^2} + 2\,\frac{(m-1)\left(\frac{\partial^2}{\partial r^2}u_1(r,\theta,z)\right)}{m-2}\\
&\quad + 2\,\frac{(m-1)\left(\frac{\partial}{\partial r}u_1(r,\theta,z)\right)}{(m-2)\,r} - \frac{(3m-4)\left(\frac{\partial}{\partial\theta}u_2(r,\theta,z)\right)}{(m-2)\,r^2} - \frac{m\,u_1(r,\theta,z)}{(m-2)\,r^2}\\
&\quad + \frac{m\left(\frac{\partial^2}{\partial r\,\partial\theta}u_2(r,\theta,z)\right)}{(m-2)\,r} + \left(\frac{\partial^2}{\partial z^2}u_1(r,\theta,z)\right) - \frac{u_1(r,\theta,z)}{r^2} = 0
\end{aligned}\tag{2}$$

$$\begin{aligned}
&\left(\frac{\partial^2}{\partial r^2}u_2(r,\theta,z)\right) + \frac{m\left(\frac{\partial^2}{\partial r\,\partial\theta}u_1(r,\theta,z)\right)}{(m-2)\,r} + 2\,\frac{(m-1)\left(\frac{\partial^2}{\partial\theta^2}u_2(r,\theta,z)\right)}{(m-2)\,r^2} + \frac{\frac{\partial}{\partial r}u_2(r,\theta,z)}{r}\\
&\quad + \left(\frac{\partial^2}{\partial z^2}u_2(r,\theta,z)\right) + \frac{(3m-4)\left(\frac{\partial}{\partial\theta}u_1(r,\theta,z)\right)}{(m-2)\,r^2} + \frac{m\left(\frac{\partial^2}{\partial z\,\partial\theta}u_3(r,\theta,z)\right)}{(m-2)\,r}\\
&\quad - \frac{u_2(r,\theta,z)}{r^2} = 0
\end{aligned}\tag{3}$$

$$\frac{\frac{\partial}{\partial r}u_3(r,\theta,z)}{r} + \frac{\frac{\partial^2}{\partial\theta^2}u_3(r,\theta,z)}{r^2} + \frac{m\left(\frac{\partial^2}{\partial z\,\partial r}u_1(r,\theta,z)\right)}{m-2} + 2\,\frac{(m-1)\left(\frac{\partial^2}{\partial z^2}u_3(r,\theta,z)\right)}{m-2}$$

$$+\left(\frac{\partial^2}{\partial r^2}u_3(r,\theta,z)\right)+\frac{m\left(\frac{\partial^2}{\partial z\,\partial\theta}u_2(r,\theta,z)\right)}{(m-2)\,r}+\frac{m\left(\frac{\partial}{\partial z}u_1(r,\theta,z)\right)}{(m-2)\,r}=0 \tag{4}$$

while the stresses are

$$\sigma_{rr}=\left(2\frac{\mu}{m-2}+2\mu\right)\left(\frac{\partial}{\partial r}u_1(r,\theta,z)\right)+2\frac{\mu\left(\frac{\partial}{\partial z}u_3(r,\theta,z)\right)}{m-2}+2\frac{\mu\left(\frac{\partial}{\partial\theta}u_2(r,\theta,z)\right)}{(m-2)\,r}$$
$$+2\frac{\mu\,u_1(r,\theta,z)}{(m-2)\,r} \tag{5}$$

$$\sigma_{r\theta}=\frac{\mu\left(\frac{\partial}{\partial\theta}u_1(r,\theta,z)\right)}{r}+\mu\left(\frac{\partial}{\partial r}u_2(r,\theta,z)\right)-\frac{\mu\,u_2(r,\theta,z)}{r}$$

$$\sigma_{rz}=\mu\left(\frac{\partial}{\partial r}u_3(r,\theta,z)\right)+\mu\left(\frac{\partial}{\partial z}u_1(r,\theta,z)\right)$$

$$\sigma_{\theta\theta}=2\frac{\mu\left(\frac{\partial}{\partial r}u_1(r,\theta,z)\right)}{m-2}+2\frac{\mu\left(\frac{\partial}{\partial z}u_3(r,\theta,z)\right)}{m-2}+\left(2\frac{\mu}{r}+2\frac{\mu}{(m-2)\,r}\right)\left(\frac{\partial}{\partial\theta}u_2(r,\theta,z)\right)$$
$$+2\frac{\mu\,u_1(r,\theta,z)}{(m-2)\,r}+2\frac{\mu\,u_1(r,\theta,z)}{r} \tag{6}$$

$$\sigma_{\theta z}=\frac{\mu\left(\frac{\partial}{\partial\theta}u_3(r,\theta,z)\right)}{r}+\mu\left(\frac{\partial}{\partial z}u_2(r,\theta,z)\right)$$

$$\sigma_{zz}=2\frac{\mu\left(\frac{\partial}{\partial r}u_1(r,\theta,z)\right)}{m-2}+2\frac{\mu\left(\frac{\partial}{\partial\theta}u_2(r,\theta,z)\right)}{(m-2)\,r}+\left(2\frac{\mu}{m-2}+2\mu\right)\left(\frac{\partial}{\partial z}u_3(r,\theta,z)\right)$$
$$+2\frac{\mu\,u_1(r,\theta,z)}{(m-2)\,r} \tag{7}$$

By using a Fourier Series, this 3D problem can be reduced to a sequence of 2D problems. Specifically, the substitutions

$$u_1(r,\theta,z) = a_{0,1}(r,z)+\sum_{n=1}^{\infty}a_{n,1}(r,z)\cos\left(\frac{n\theta}{2}\right) \tag{8}$$

$$u_2(r,\theta,z) = \sum_{n=1}^{\infty}b_{n,2}(r,z)\sin\left(\frac{n\theta}{2}\right) \tag{9}$$

$$u_3(r,\theta,z) = a_{0,3}(r,z)+\sum_{n=1}^{\infty}a_{n,3}(r,z)\cos\left(\frac{n\theta}{2}\right) \tag{10}$$

allow us to now solve slightly reduced equations independent of θ, to be solved on the domain shown in figure 2.

Solution of the problem now requires a domain decomposition into 4 domains as indicated. By symmetry, the top and bottom portions are equal. We thus have a 2-dimensional, 3 unknown, 4 domain problem. The boundary conditions for the numbered regions are shown in table 1; for the primed regions, $u_{z'}=-u_z$, $u_{\theta'}=u_\theta$ and $u_{z'}=u_z$. From asymptotic expansions, it is known that the stresses, hence derivatives of displacements, are unbounded at the lower and upper right corners of domain 2, making this problem well suited for sinc methods but not finite elements or finite differences.

Boundary	Region 1	Region 2
left	$\frac{\partial}{\partial \theta}\mathbf{u} = 0$	$\frac{\partial}{\partial \theta}\mathbf{u} = 0$
right	$\sigma_{rr} = \sigma_{r\theta} = \sigma_{rz} = 0$	$\sigma_{rr}^{(2)} = \sigma_{rr}^{(3)}, \ \sigma_{r\theta}^{(2)} = \sigma_{r\theta}^{(3)}, \ \sigma_{rz}^{(2)} = \sigma_{rz}^{(3)}$
top	$\mathbf{u}_1 = \mathbf{u}_2$, $\frac{\partial}{\partial z}\mathbf{u}_1 = \frac{\partial}{\partial z}\mathbf{u}_2$	$\sigma_{rz} = \sigma_{\theta z} = \sigma_{zz} = 0$
bottom	$\mathbf{u}_1 = \mathbf{u}_{1'}$, $\frac{\partial}{\partial z}\mathbf{u}_1 = \frac{\partial}{\partial z}\mathbf{u}_{1'}$	see 1, top

Boundary	Region 3	Region 4
left	see 2, right	$\sigma_{rr} = \sigma_{r\theta} = \sigma_{rz} = 0$
right	$\sigma_{rr}, \ \sigma_{r\theta}, \ \sigma_{rz}$ specified	see 3, right
top	$\sigma_{rz} = \sigma_{\theta z} = \sigma_{zz} = 0$	see 2, top
bottom	$\mathbf{u}_3 = \mathbf{u}_4$, $\frac{\partial}{\partial z}\mathbf{u}_3 = \frac{\partial}{\partial z}\mathbf{u}_4$	see 1, bottom

Table 1: Boundary conditions for reduced problem.

2 Solution Approach

Since the problem is known to have singularities, and since both finite element and -difference methods are known to have significant difficulties on it, considering a new numerical approach is of interest. Here, we consider sinc-collocation methods for PDE's. The solution will be series based on functions of the form

$$\frac{\sin(\pi x)}{\pi x},$$

called *Sinc* functions, and where necessary will be complemented with cubic splines. For interpolating functions of class L_α and their derivatives, the series

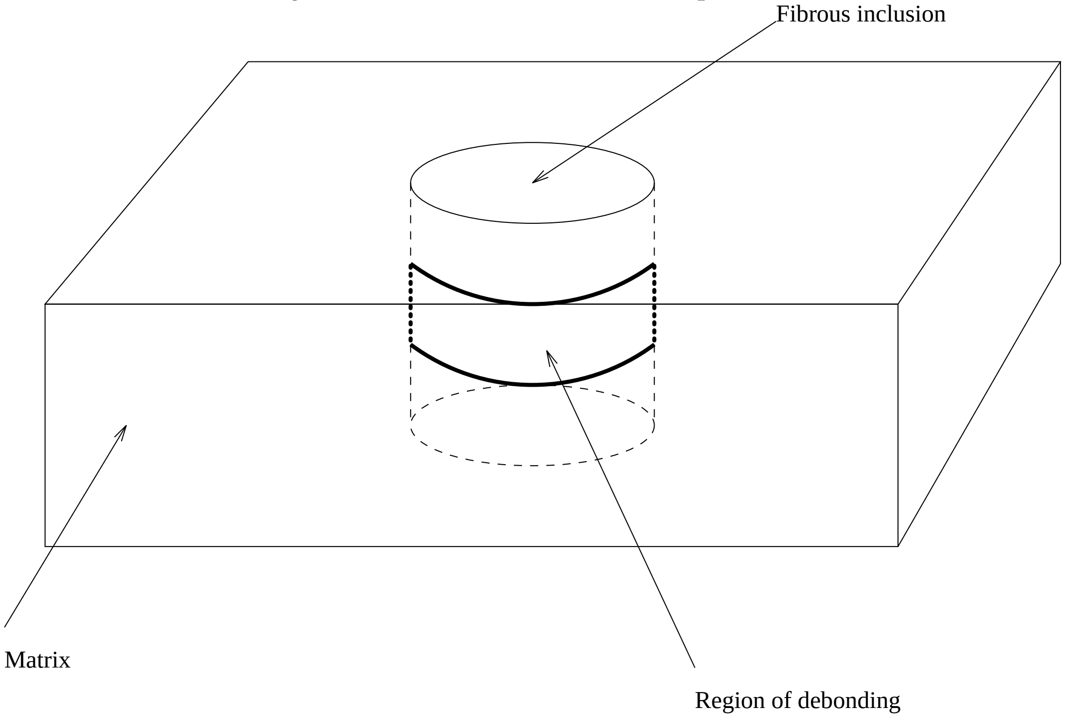
$$y_m(x) = \sum_{k=-M}^{N} \frac{F(\psi(kh))}{g(\psi(kh))} \text{sinc}((\phi(x) - kh)/h) g(x) \tag{11}$$

will be used. Here,

- $\phi(x) = \ln(\frac{x}{1-x})$ is a conformal map from [0,1] to $(-\infty, \infty)$
- $\psi(x) = \phi^{-1}(x) = \frac{e^x}{e^x+1}$
- $m = N + M + 1$
- $F(x)$ is the function to be interpolated.
- h is the grid spacing, usually picked as $h = \pi/\sqrt{\alpha N}$, where α depends on the endpoint behavior of $F(x)$.

The nullifier function $g(x)$ in (11) influences behavior at the endpoints, and will be needed later.

Figure 1: The domain for the full 3D problem.



This series can very accurately approximate bounded functions with bounded derivatives, like $f = x^{3/2}(1-x), x \in [0,1]$. In a number of tests, the interpolation of bounded functions with unbounded derivatives also gave very good results, e.g. $f = x^{1/2}(1-x), x \in [0,1]$ with a relative error of less than 2% in function and derivative.

This accuracy cannot easily be achieved with polynomials (if at all), and finite differences won't work either, making further investigation of sinc methods worthwhile.

In the following, we present a chronological summary of the results achieved to date, and what remains to be done to get the main problem.

2.1 1D Interpolation

2.2 1D Boundary Value Problems

Several single domain problems were coded and tested, along with some multiple-domain problems. As a representative example, consider the problem

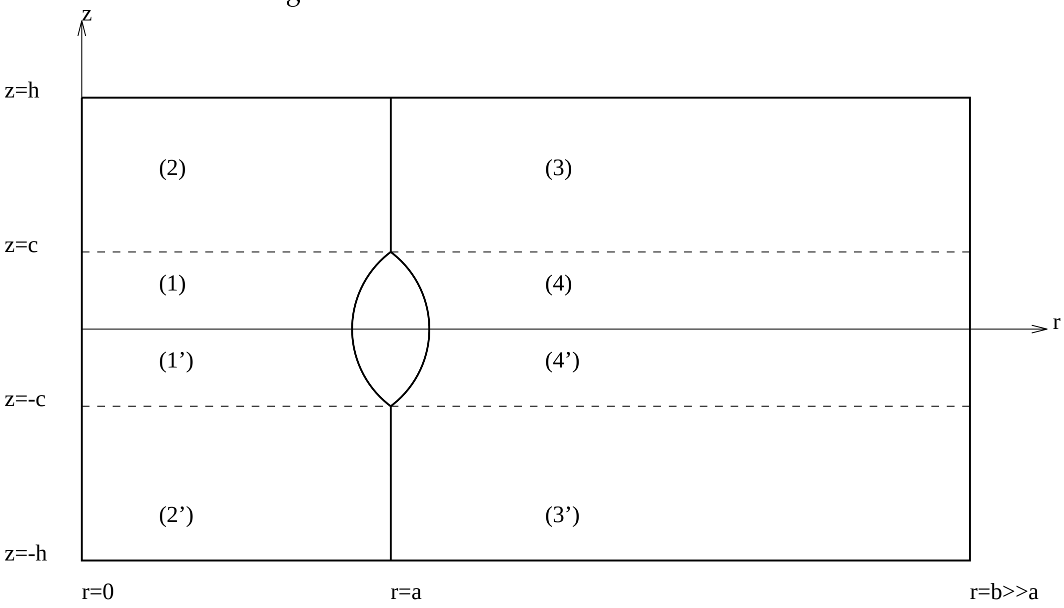
$$(Lu)(x) = -u''(x) + p(x)u'(x) + q(x) = f(x), \tag{12}$$

$$u(0) = a \tag{13}$$

$$u'(1) = b \tag{14}$$

for appropriate choices of $p(x)$ and $q(x)$ Notice that mixed conditions of the form $a_0 u(0) + a_1 u'(0) = a$ are handled similarly.

Figure 2: The domain after series substitution.



As for interpolation, first re-map the equation with an appropriately chosen $\eta(x)$ so that the equation $Lv = \tilde{f}(x)$ formed by the substitution $v(x) = u(x) + \eta(x)$ has homogeneous boundary conditions. Next, assume a solution of the form

$$v_m(x) = c_{-N-1}\omega_a + \sum_{k=-M}^{N} c_k \text{sinc}((\phi(x) - kh)/h)g(x) + c_{M+1}\omega_b \tag{15}$$

where ω_a and ω_b are cubic splines satisfying the homogeneous boundary conditions, and $g(x)$ the weight function chosen as in interpolation.

Notice this solution already satisfies boundary conditions; all that remains is to pick the c_i to satisfy the equation (12). Substituting the assumed form (15) into (12) and evaluating at the $2N+3$ sinc points $z_k = \psi(kh), k = -N-1..N+1$ yields a square system which can be solved for the c_i.

Several tests were run; some results are shown in figure 3 for $a = 2, b = 1, p(x) = -\frac{1}{6\,x}$, $q(x) = x^{-2}$ and

$$\begin{aligned} f(x) &= -\frac{-57\,x^3 + 66\,x^4 + 42\,x^{3/2} - 10\,x^{3/2}\sqrt{1-x} - 24\sqrt{x}}{12\,x^{5/2}\sqrt{1-x}} - 14 + 30\,x \\ &\quad + \frac{-14\,x + 15\,x^2}{6\,x} - \frac{2 - 7\,x^2 + 5\,x^3}{x^2} \end{aligned}$$

This problem has exact answer

$$u(x) = \left(2 + x^{3/2}\right)(1-x)^{3/2} + x$$

Some Notes:

1. This approach of satisfying boundary conditions with splines prior to solving a system does not extend to multiple unknowns; it also makes coding much more complicated.
2. Including all possibly needed splines and making the boundary conditions part of the collocation matrix is much easier to implement, and convergence is better.
3. This problem was also done using a combination of MAPLE and C++; here again, the approach in step 2 would have been preferable...

2.3 2D single unknown test problem

The test problem is straightforward:

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \text{ on } [0,1] \times [0,\pi] \tag{16}$$

$$u(1,\theta) = \sin(\theta/2), \;\; u_\theta(r,\pi) = 0 \tag{17}$$

$$u_\theta(0,\theta) = 0, \;\; u(r,0) = 0 \tag{18}$$

This is a modification of a half plane problem; the solution is

$$u(r,\theta) = \sqrt{r}\sin(\theta/2) \tag{19}$$

The sinc functions will be simple product schemes on the square. They satisfy homogeneous boundary conditions[1] , and thus require splines for completeness and to satisfy nontrivial boundary conditions. For u we thus have an assumed solution of the form[2]

$$\begin{aligned} u(x,y) \;\; = \;\; & u_{\text{left\_value}}(x,y) + u_{\text{right\_value}}(x,y) \\ & + u_{\text{top\_value}}(x,y) + u_{\text{top\_value}}(x,y) \\ & + u_{\text{bottom\_value}}(x,y) + u_{\text{left\_partial}}(x,y) \\ & + u_{\text{right\_partial}}(x,y) + u_{\text{top\_partial}}(x,y) \\ & + u_{\text{bottom\_partial}}(x,y) + u_{\text{corner}}(x,y) + u_{\text{sinc}}(x,y) \end{aligned} \tag{20}$$

where these parts are defined as

$$u_{\text{left\_value}}(x,y) \;\; = \;\; \sum_{i=-M_2+1}^{N_2-1} \text{sinc}\left(\frac{\phi_2(y) - ih_2}{h_2}\right) g_2(y)\mathcal{S}_{(1,0,0,0,0,1)}(x) \tag{21}$$

$$u_{\text{right\_value}}(x,y) \;\; = \;\; \sum_{i=-M_2+1}^{N_2-1} \text{sinc}\left(\frac{\phi_2(y) - ih_2}{h_2}\right) g_2(y)\mathcal{S}_{(0,0,1,0,0,1)}(x) \tag{22}$$

$$u_{\text{top\_value}}(x,y) \;\; = \;\; \sum_{j=-M_1+1}^{N_1-1} \text{sinc}\left(\frac{\phi_1(x) - jh_1}{h_1}\right) g_1(x)\mathcal{S}_{(0,0,1,0,0,\pi)}(y) \tag{23}$$

[1] Without weight function, $u(x,0) = u(x,1) = u(0,y) = u(1,y)$; with weight function, the derivatives also vanish.

[2] Finding suitable splines, and the associated programming, took a while: from [Jan-2-1995] to [Feb-5-1995]; this version is s7. Refining the code to make extensions to multiple unknowns smooth required another 3 days. The approach presented here is the result. This approach resulted in 6700 total lines of code, of which 3800 were generated through MAPLE.

$$u_{\text{bottom\_value}}(x,y) = \sum_{j=-M_1+1}^{N_1-1} \text{sinc}\left(\frac{\phi_1(x)-jh_1}{h_1}\right) g_1(x)\mathcal{S}_{(1,0,0,0,0,\pi)}(y) \tag{24}$$

$$u_{\text{left\_partial}}(x,y) = \sum_{i=-M_2+1}^{N_2-1} \text{sinc}\left(\frac{\phi_2(y)-ih_2}{h_2}\right) g_2(y)\mathcal{S}_{(0,1,0,0,0,1)}(x) \tag{25}$$

$$u_{\text{right\_partial}}(x,y) = \sum_{i=-M_2+1}^{N_2-1} \text{sinc}\left(\frac{\phi_2(y)-ih_2}{h_2}\right) g_2(y)\mathcal{S}_{(0,0,0,1,0,1)}(x) \tag{26}$$

$$u_{\text{top\_partial}}(x,y) = \sum_{j=-M_1+1}^{N_1-1} \text{sinc}\left(\frac{\phi_1(x)-jh_1}{h_1}\right) g_1(x)\mathcal{S}_{(0,0,0,1,0,\pi)}(y) \tag{27}$$

$$u_{\text{bottom\_partial}}(x,y) = \sum_{j=-M_1+1}^{N_1-1} \text{sinc}\left(\frac{\phi_1(x)-jh_1}{h_1}\right) g_1(x)\mathcal{S}_{(0,1,0,0,0,\pi)}(y) \tag{28}$$

$$u_{\text{corner}}(x,y) = \sum_{i=1}^{4}\sum_{j=1}^{4} \mathcal{V}_j(y)\mathcal{H}_i(x) \tag{29}$$

$$u_{\text{sinc}}(x,y) = \sum_{i=-M_2}^{N_2}\sum_{j=-M_1}^{N_1} d_{ij}\text{sinc}\left(\frac{\phi_2(y)-ih_2}{h_2}\right) g_2(y)\text{sinc}\left(\frac{\phi_1(x)-jh_1}{h_1}\right) g_1(x) \tag{30}$$

(31)

where the $\mathcal{V}_j(y)$ and $\mathcal{H}_i(x)$ are enumerations:

$$\begin{array}{ll} \mathcal{V}_1 = \mathcal{S}_{(0,0,1,0,0,\pi)}(y) & \mathcal{H}_1 = \mathcal{S}_{(0,0,1,0,0,1)}(x) \\ \mathcal{V}_2 = \mathcal{S}_{(0,0,0,1,0,\pi)}(y) & \mathcal{H}_2 = \mathcal{S}_{(0,0,0,1,0,1)}(x) \\ \mathcal{V}_3 = \mathcal{S}_{(1,0,0,0,0,\pi)}(y) & \mathcal{H}_3 = \mathcal{S}_{(1,0,0,0,0,1)}(x) \\ \mathcal{V}_4 = \mathcal{S}_{(0,1,0,0,0,\pi)}(y) & \mathcal{H}_4 = \mathcal{S}_{(0,1,0,0,0,1)}(x) \end{array}$$

Also, from here on, $x = r$ and $y = \theta$.

The form in (20) allows for all boundary conditions, and so the boundary conditions are included as part of the collocation matrix. Although the problem of this section can be (and has been) solved with simpler forms that automatically satisfy the boundary conditions, this form will directly extend to multiple-domain, multiple-unknown problems. Using this form, some exact and computed answers are shown in figures 4, 6, 5, 9, 8 and 7.

2.4 2D, 2 unknown tests

Using the above basis, numerous tests were run on two unknown problems. A typical example is the following.

For the stresses σ_{xx}, σ_{xy} and σ_{yy} defined by

$$\sigma_{xx} = (\lambda + 2\,G)\,\frac{d}{dx}u + \lambda\,\frac{d}{dy}v \tag{32}$$

$$\sigma_{xy} = G\left(\frac{d}{dy}u + \frac{d}{dx}v\right) \tag{33}$$

$$\sigma_{yy} = (\lambda + 2\,G)\,\frac{d}{dy}v + \lambda\,\frac{d}{dx}u \tag{34}$$

we solve the equations

$$G\left(\frac{d^2}{dx^2}u + \frac{d^2}{dy^2}u\right) + (\lambda + G)\left(\frac{d^2}{dx^2}u + \frac{d^2}{dx\,dy}v\right) = -X\_1 \tag{35}$$

and

$$G\left(\frac{d^2}{dx^2}v + \frac{d^2}{dy^2}v\right) + (\lambda + G)\left(\frac{d^2}{dy\,dx}u + \frac{d^2}{dy^2}v\right) = -X\_2 \tag{36}$$

subject to the boundary conditions

$$\sigma_{yy} = f_3(x) \tag{37}$$

$$\sigma_{xy} = g_3(x) \tag{38}$$

on the top,

$$\sigma_{xx} = f_2(y) \tag{39}$$

$$\sigma_{xy} = g_2(y) \tag{40}$$

on the left,

$$\sigma_{yy} = f_4(x) \tag{41}$$

$$\sigma_{xy} = g_4(x) \tag{42}$$

on the bottom, and

$$u = f_1(y) \tag{43}$$

$$v = g_1(y) \tag{44}$$

on the right. Here, we picked the exact answers

$$u(x,y) = (\lambda + G)\cos(y)\,(1-x)^{3/2} \tag{45}$$

$$v(x,y) = (\lambda + G)\,y^{3/2}\,(1-x)^{\frac{11}{10}} \tag{46}$$

and determined the f's and g's. The resulting problem was then solved using the sinc basis and collocation. Some results for $M = N = 8$ are shown in figures 10 and 11.

2.5 2D, 2 unknown, 2 domain tests

2.5.1 Test problems with bounded derivatives

Numerous problems were run with bounded first derivatives in the answer, and various types of boundary conditions.

As one example, we have the following. For σ_{xx}, σ_{xy}, and σ_{yy} given by

$$\sigma_{xx} = (\,\lambda + 2\,\mu\,)\,\left(\frac{\partial}{\partial r}\,u\right) + \lambda\,\left(\frac{\partial}{\partial z}\,w\right) + \frac{\lambda\,u}{r},$$

$$\sigma_{xy} = \mu \left(\frac{\partial}{\partial r} w \right) + \mu \left(\frac{\partial}{\partial z} u \right)$$

and

$$\sigma_{yy} = \lambda \left(\frac{\partial}{\partial r} u \right) + (\lambda + 2\mu) \left(\frac{\partial}{\partial z} w \right) + \frac{\lambda u}{r}$$

respectively, solve the equations

$$L1 = \frac{m \left(\frac{\partial^2}{\partial y \, \partial x} u \right)}{m-2} + \frac{\frac{\partial}{\partial x} w}{x} + 2 \frac{(m-1) \left(\frac{\partial^2}{\partial y^2} w \right)}{m-2} + \frac{m \left(\frac{\partial}{\partial y} u \right)}{(m-2)\, x} + \left(\frac{\partial^2}{\partial x^2} w \right) = 0 \quad (47)$$

and

$$L2 = \frac{m \left(\frac{\partial^2}{\partial y \, \partial x} w \right)}{m-2} + 2 \frac{(m-1) \left(\frac{\partial^2}{\partial x^2} u \right)}{m-2} + 2 \frac{(m-1) \left(\frac{\partial}{\partial x} u \right)}{(m-2)\, x} - \frac{u}{x^2} - \frac{m u}{(m-2)\, x^2} + \left(\frac{\partial^2}{\partial y^2} u \right) = 0 \quad (48)$$

on the split domain $0 \le x \le 1, 0 \le y \le 1$ and $1 \le x \le 2, 0 \le y \le 1$, with different m for each domain. For the left region's boundary conditions, specify σ_{yy} and σ_{xy} on top, u and w on the left and bottom. For the right region, specify σ_{yy} and σ_{xy} on the top and bottom, and u and w on the right. For the connecting boundary, match the function values and normal derivatives; thus, $u_1 = u_2$, $w_1 = w_2$, $\partial u_1/\partial x = \partial u_2/\partial x$ and $\partial w_1/\partial x = \partial w_2/\partial x$.

Here, we picked $\mu = 11.5$, $\lambda = 15.88$ and $m = 3.448$ for region 1, $\mu = 5.6$, $\lambda = 10.87$ and $m = 3.030$ for region 2. For both regions, $h = \pi/\sqrt{2N}$ with $N = 12$, and the exact answers are taken to be

$$u = \left((x-1)^2 + (y-1)^2 \right)^{2/3} \quad (49)$$

and

$$w = \left((x-2)^2 + y^2 \right)^{2/3} \quad (50)$$

The computed results are compared to the exact answers in figures 12, 13, 14, and 15.

For other test problems with bounded first derivatives and with comparable magnitude between their unknowns, very good results were also obtained.

2.5.2 Test problems with unbounded derivatives

Interpolation Test Anticipating problems with boundary conditions involving unbounded derivatives, we first tested the quality of approximation of function and derivatives after *interpolation* of the function

$$f(x, y) = \rho(1-x)(1-y) - \sqrt{y}(1-x)(1-y) \quad (51)$$

with ρ defined by

$$\rho = r^{\frac{1}{2}} \sin(\theta) = \frac{y}{(x^2 + y^2)^{1/4}} \quad (52)$$

This f has properties very similar to those given by an asymptotic expansion of the solution of the main problem. The results of this interpolation were very good.

Dirichlet Problem Since these results were promising, a (mostly) dirichlet problem was also tested. Solving equations (47) and (48) on the domains $0 \leq x \leq 1$, $0 \leq y \leq 1$ and $1 \leq x \leq 20$, $0 \leq y \leq 1$ with specified values of u and w on all exterior boundaries, and requiring matching of values and normal derivatives on the interior boundary (including any necessary offset), we picked the exact solutions as

$$u = \frac{y-1}{\sqrt[4]{(x-1)^2 + (y-1)^2}} \tag{53}$$

and

$$w = (y) * (x-1) \tag{54}$$

for region 1, and

$$u = (y-1)^2 x \tag{55}$$

and

$$w = \frac{y-1}{\sqrt[4]{(x-1)^2 + (y-1)^2}} \tag{56}$$

for region 2. For the parameters $m = 3.448$, $N = 11$ for region 1 and $m = 3.030$, $N = 11$ for region 2, the results for the strip $0 \leq x \leq 13.3$, $0.70 \leq y \leq 0.99$ are shown in figures 16, 18, 17 and 19.

Notice that along the interior boundary, $\partial u/\partial x$ and $\partial w/\partial x$ are bounded, so the basis used in (20) is sufficient for this problem.

Mixed Problems – direct solution solving equations (47) and (48) with mixed boundary conditions and using the basis in equations (20) gives useless results, regardless of parameter choices.

Mixed Problems – solution after re-mapping By replacing the unknown u by $\hat{u} \equiv u/r^2$ where r is the distance from the singularity, and remapping all equations appropriately before solving, the solution obtained using the form in (20) converges. Unfortunately, the relative error in the vicinity of $r = 0$ is large, since the new unknowns have at least a second-order zero there. This error is retained after the original u is recovered, making the answers once again useless.

Other re-mappings of u were also tested, but failed because of one of the following reasons:

1. The re-mapping must be absolutely monotone so the differential equations can still be solved.
2. The remapping must be at least C^2 to be usable in the D.E.
3. The remapping must have one and only one zero, at least of first order, at the singularity of the sought solution.

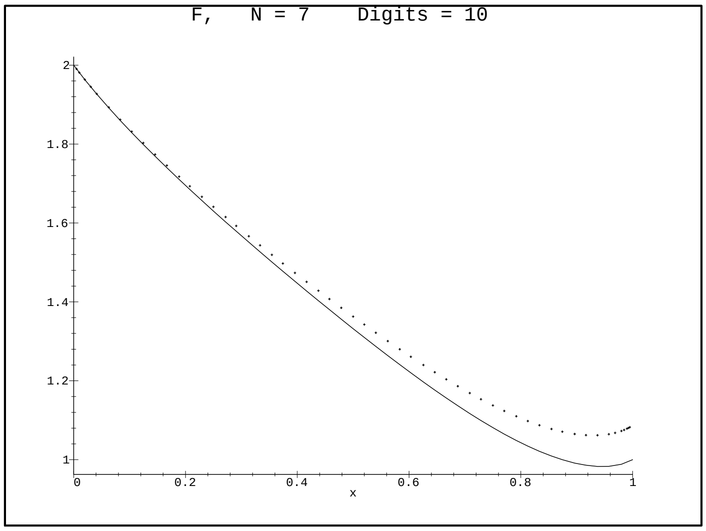
These requirements imply that any remapping used will be quadratic in a neighborhood of the singularity, and hence the numerical solution will incur a large relative error there.

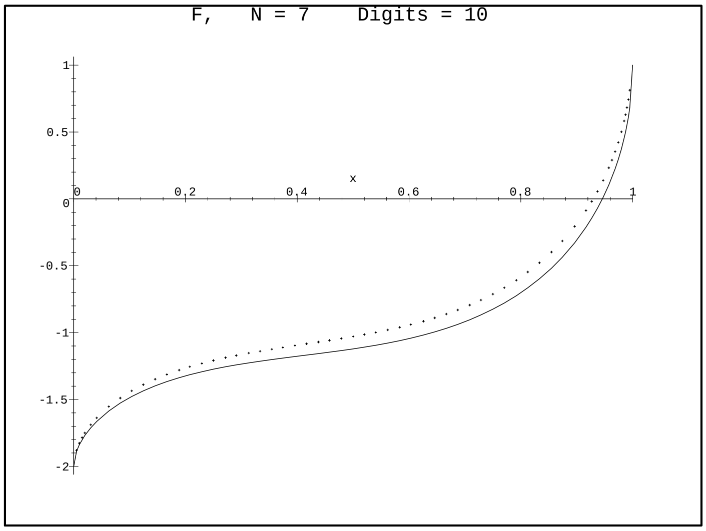
2.6 Current Work

The nullifier $g(x)$ used in the Sinc basis for the previous problems is no longer an adequate modification of the basis when the first derivatives of the sought functions are unbounded at the boundaries; in fact, using *no* nullifier (i.e., $g(x) = 1$) is the ideal choice for interpolation when no derivative conditions are used.

Having unbounded derivatives as part of the boundary conditions, as required by the main problem's matching boundaries, presents a major numerical difficulty, aside from requiring a different set of basis functions. The *sinc* basis used in (20) has bounded derivatives, and so will need to be modified to accommodate unbounded ones. To this end, some tests were done in which the derivative of a sinc series was used in collocating the unbounded derivative of a L_α function. Using this representation with appropriate splines for the parts of the solution known to be unbounded should give good results. This is currently being worked on; coding and testing of the code are as usual the greatest time sinks.

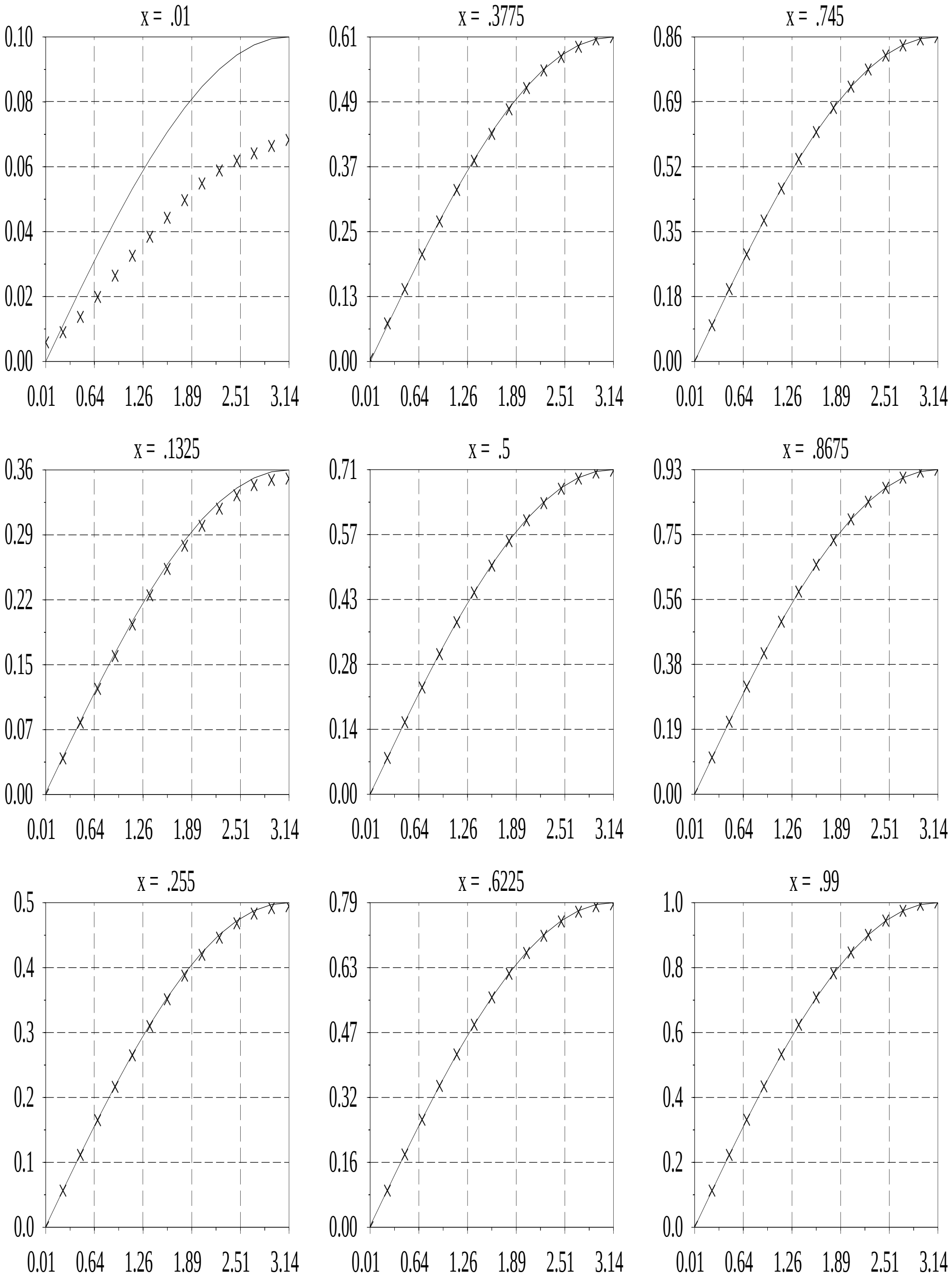
Figure 3: Comparison of F, F' and their approximations using the modified weight.





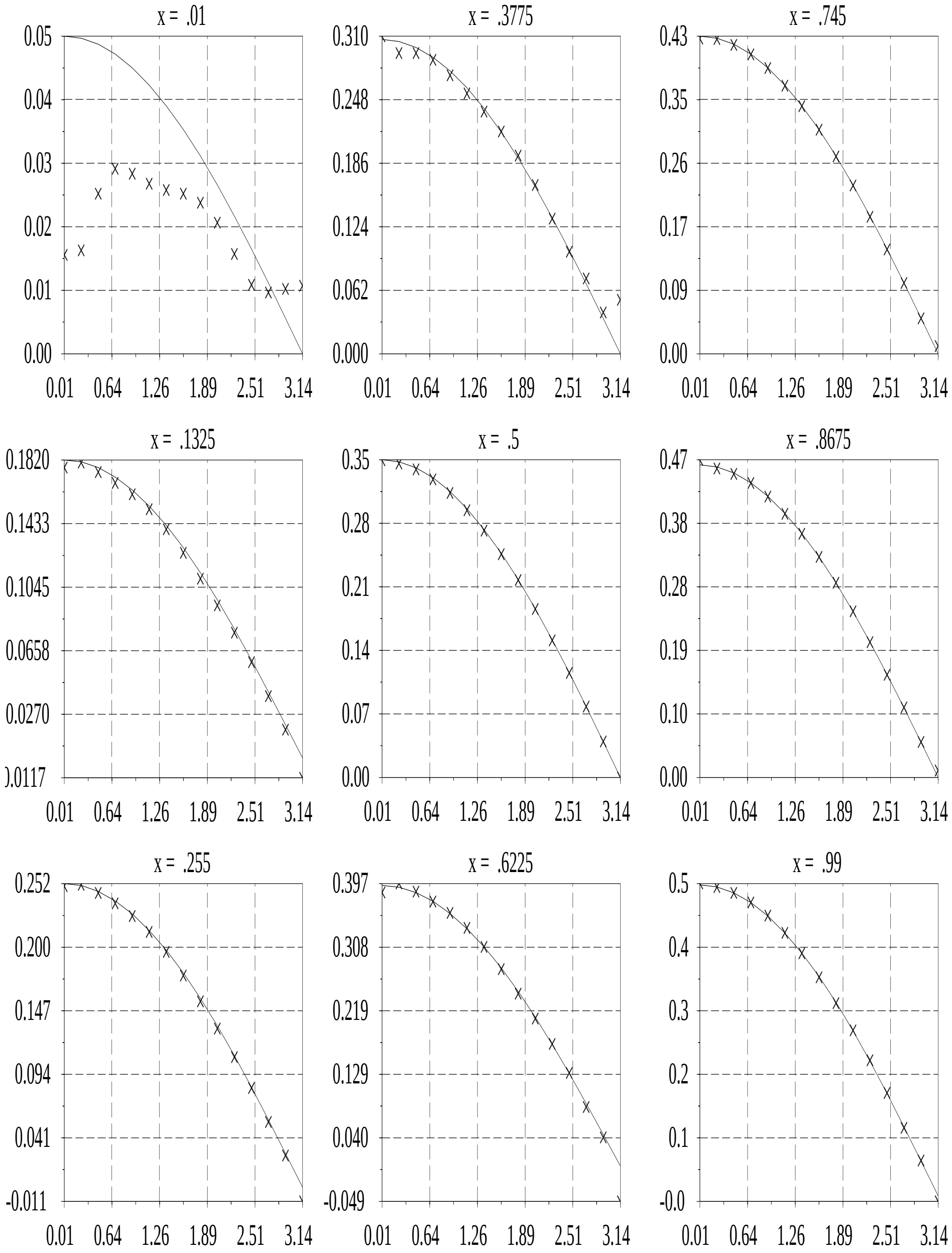
scilab_output

Figure 4: Exact and computed answers, M=N=5



scilab_output

Figure 5: Exact and computed y-partials, M=N=5.



scilab_output

Figure 6: Exact and computed x-partials, M=N=5.

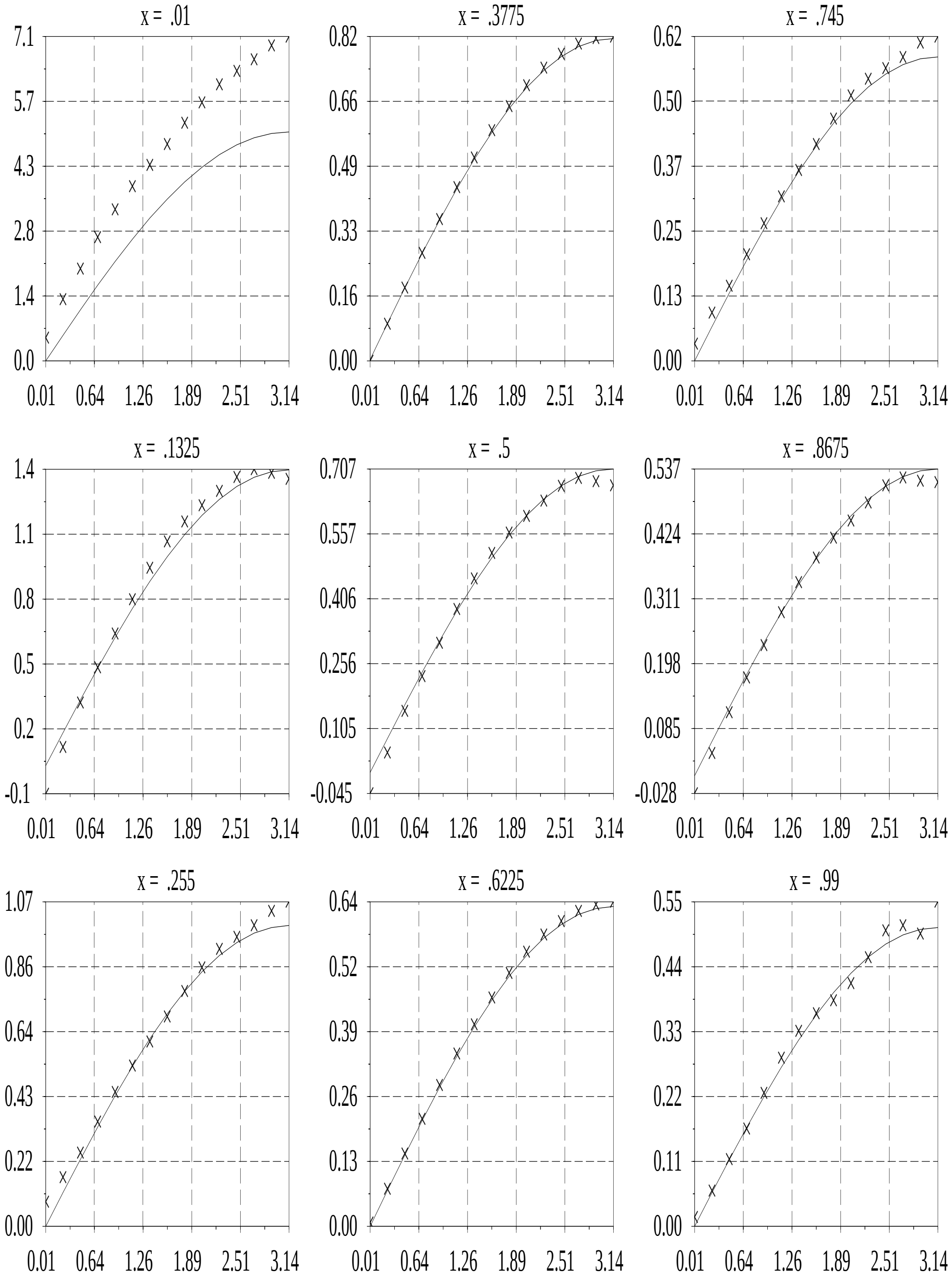
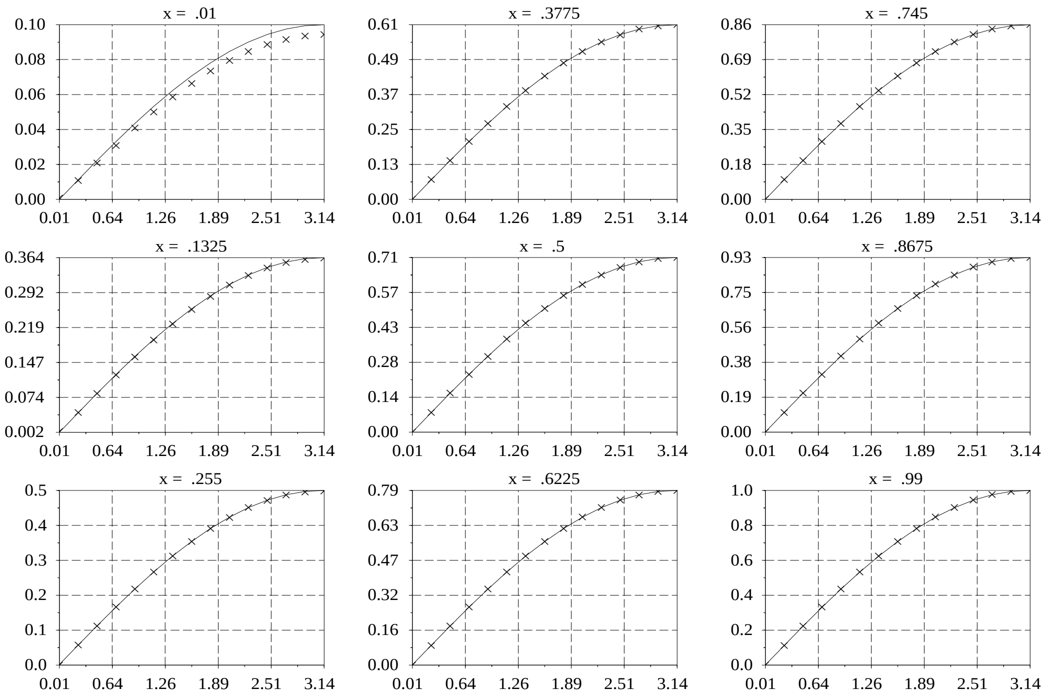
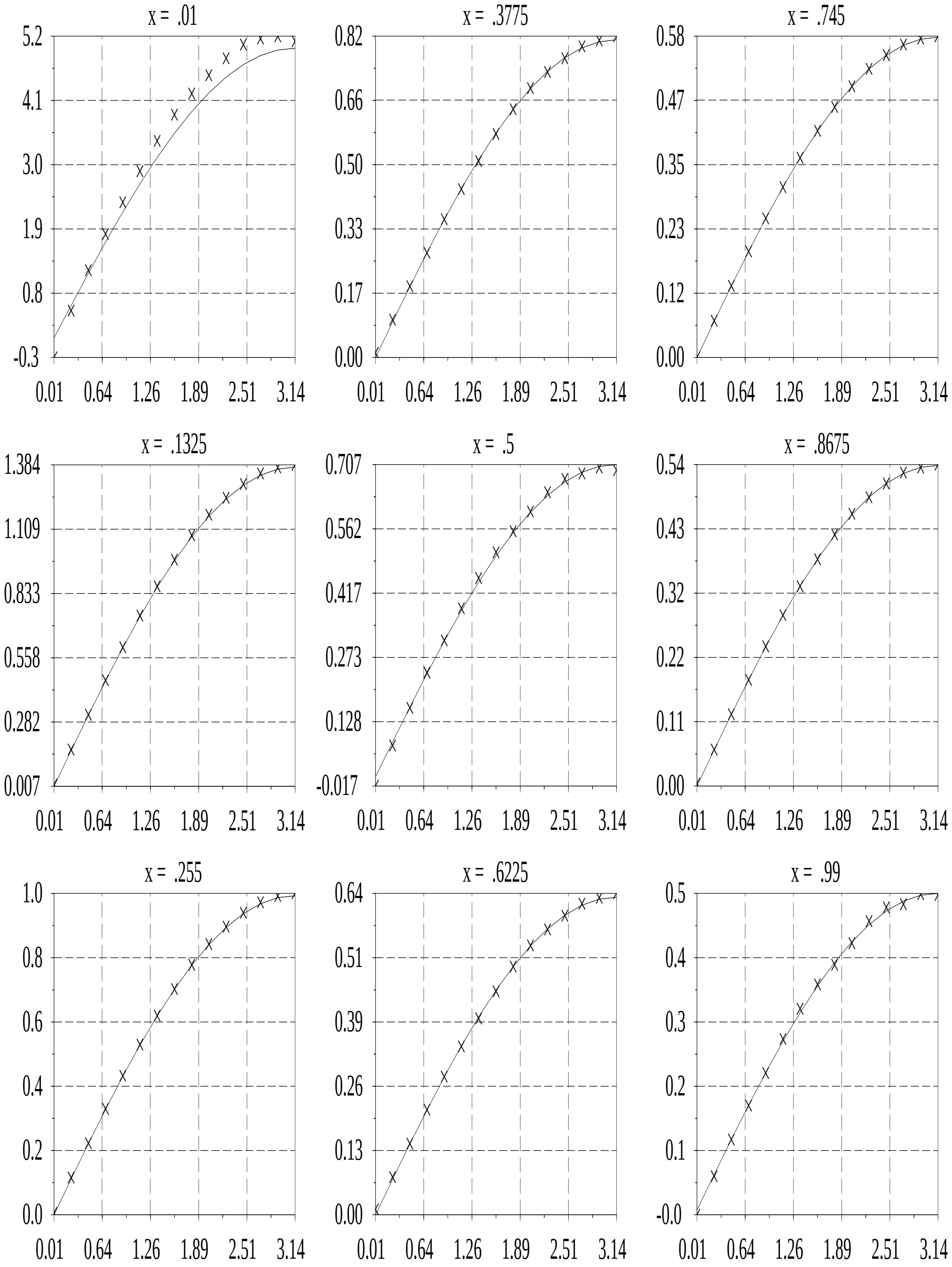


Figure 7: Exact and computed answers, M=N=9.



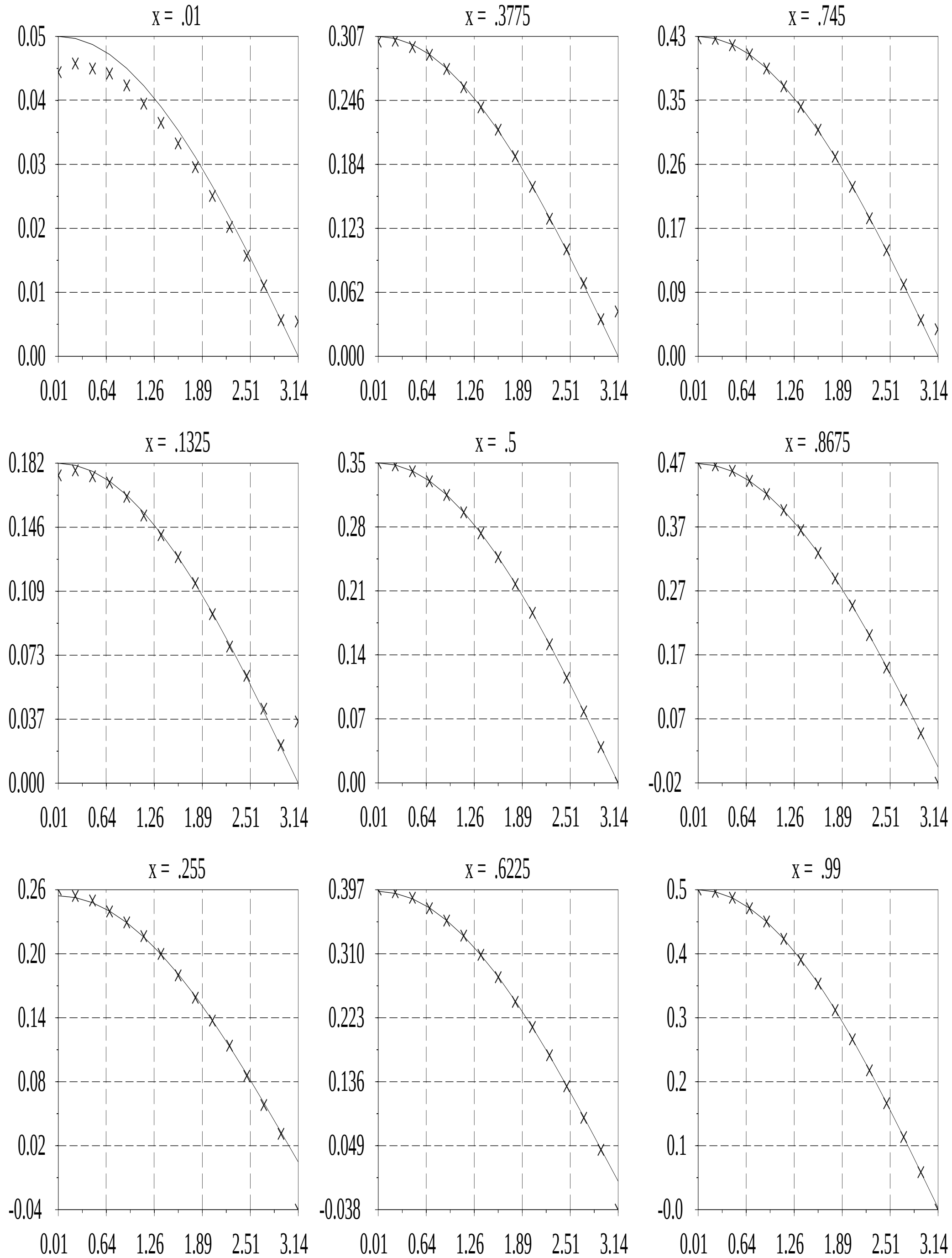
scilab_output

Figure 8: Exact and computed x-partials, M=N=9.



scilab_output

Figure 9: Exact and computed y-partials, M=N=9.



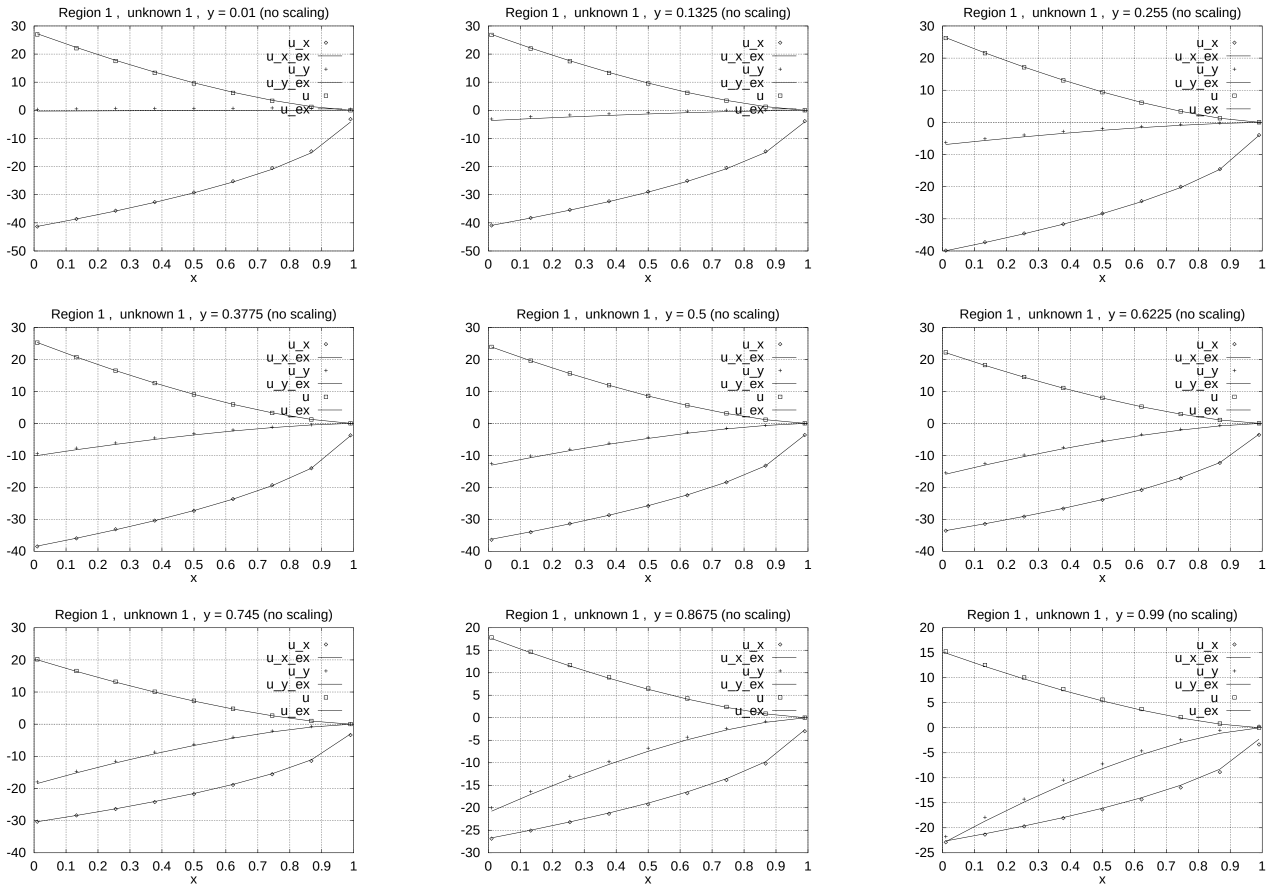


Figure 10: Comparison of exact and calculated u, u_x and u_y.

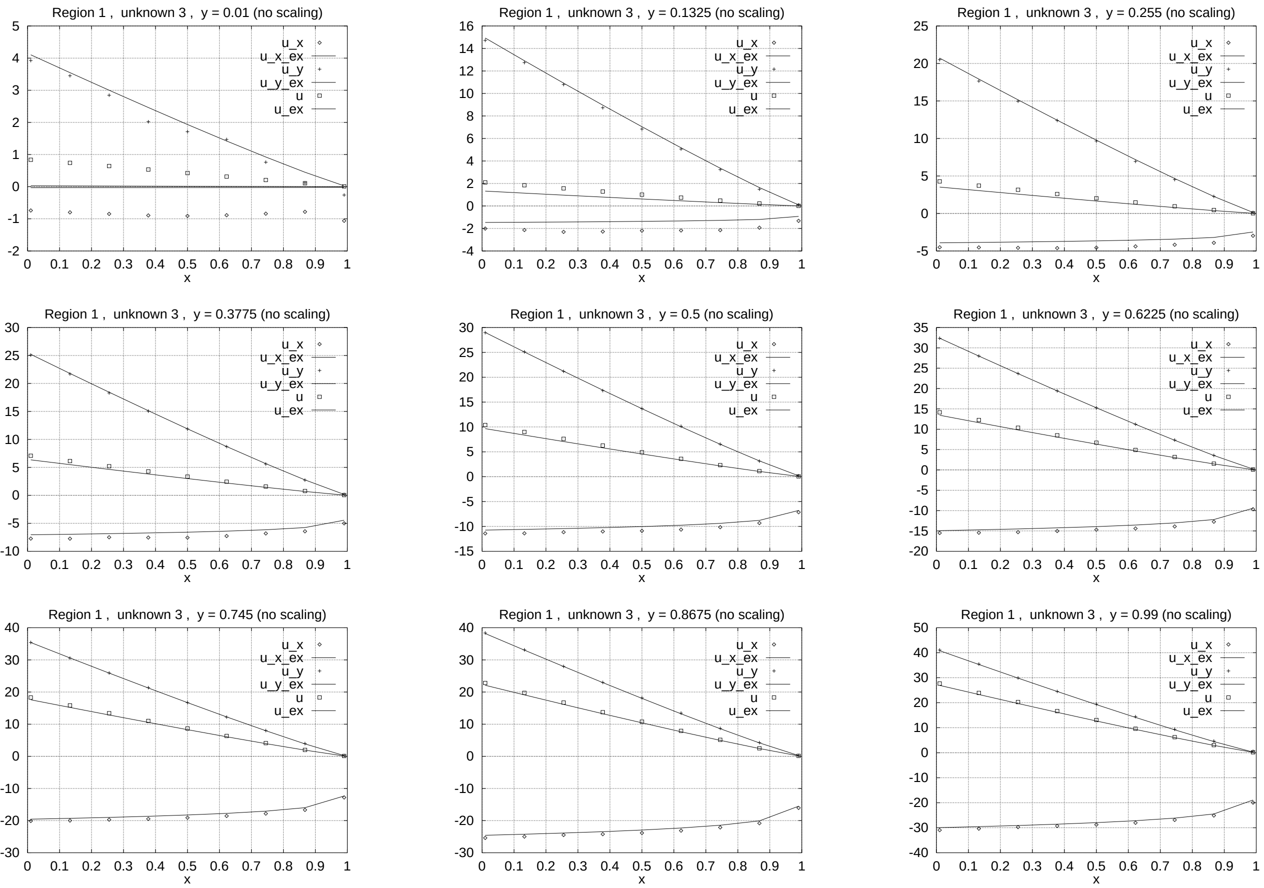


Figure 11: Comparison of exact and calculated v, v_x and v_y.

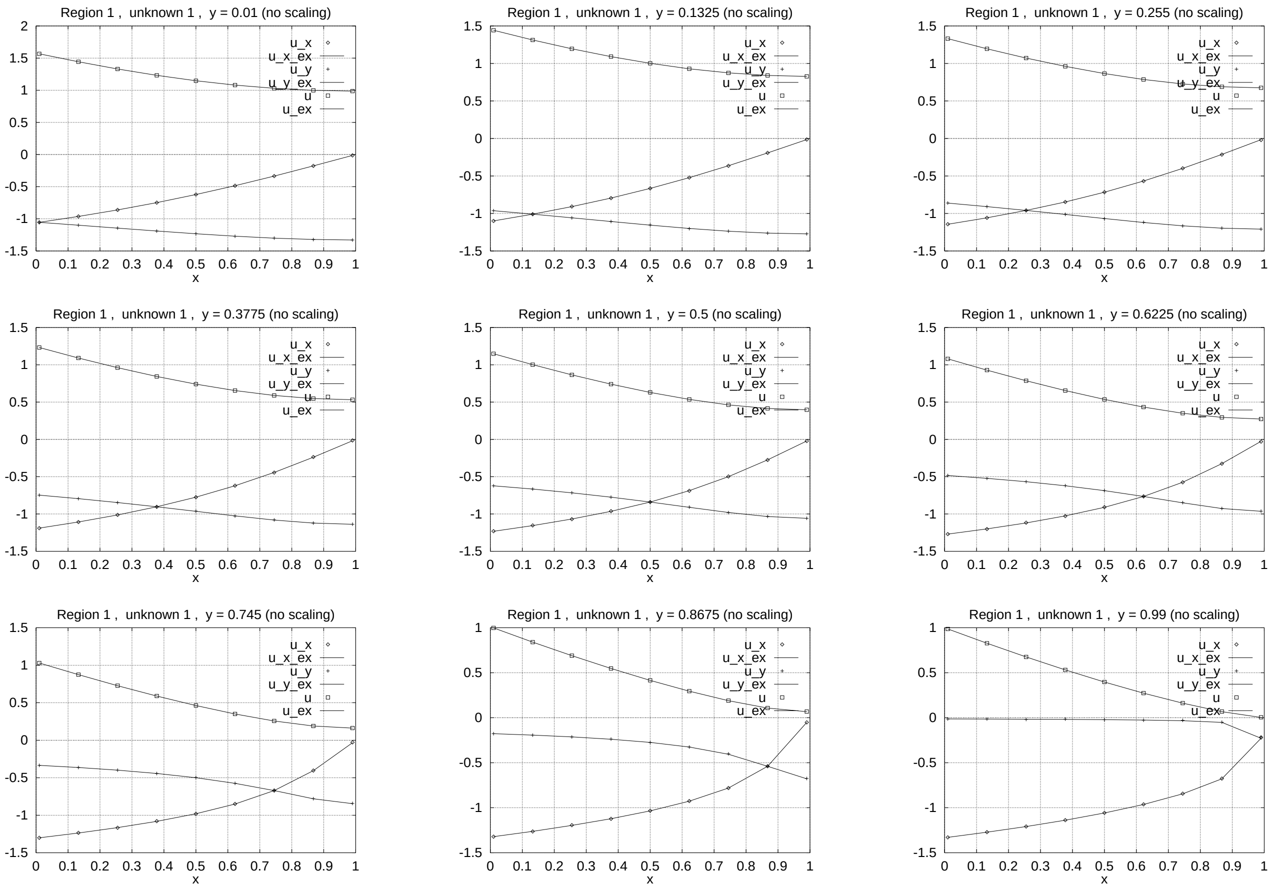


Figure 12: Comparison of exact and calculated u, u_x and u_y, region 1.

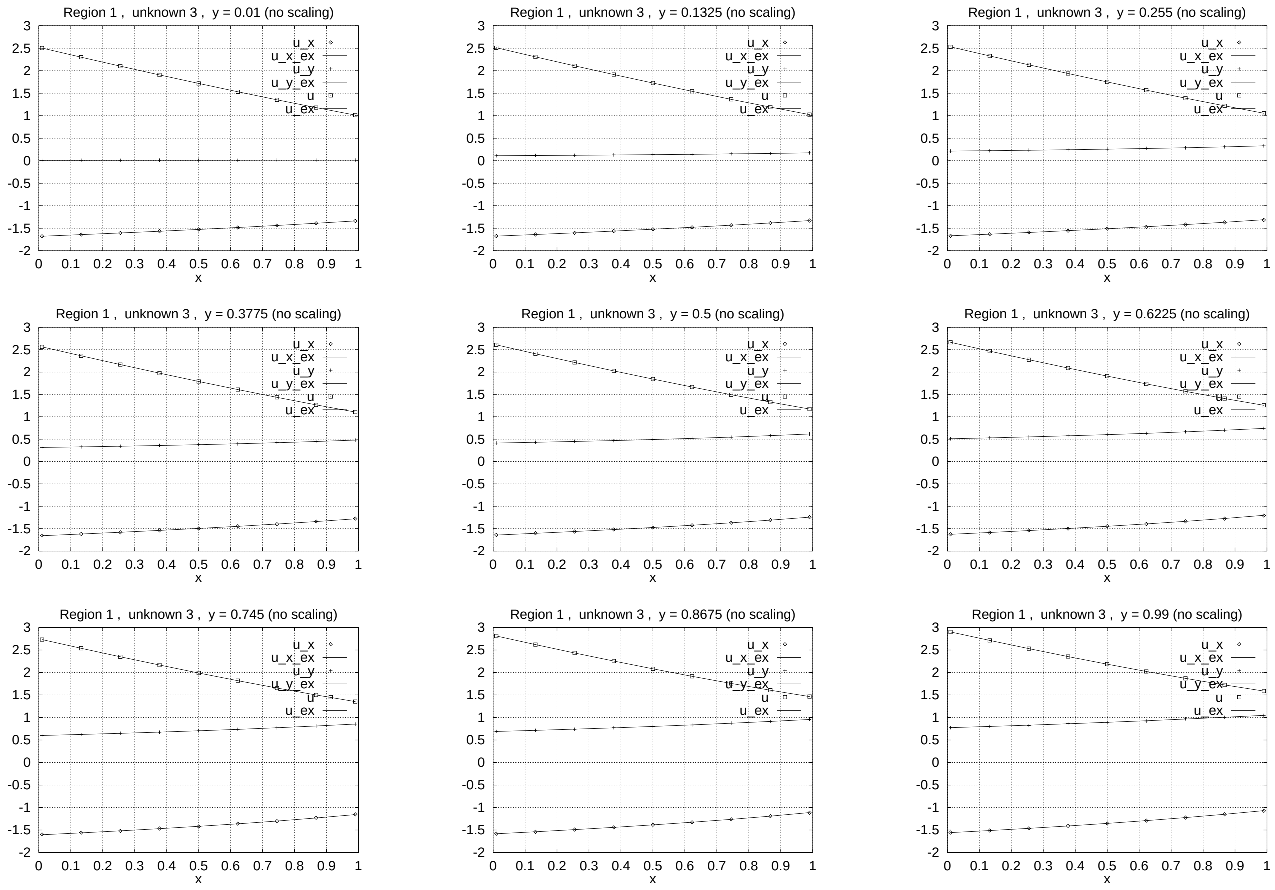


Figure 13: Comparison of exact and calculated v, v_x and v_y, region 1.

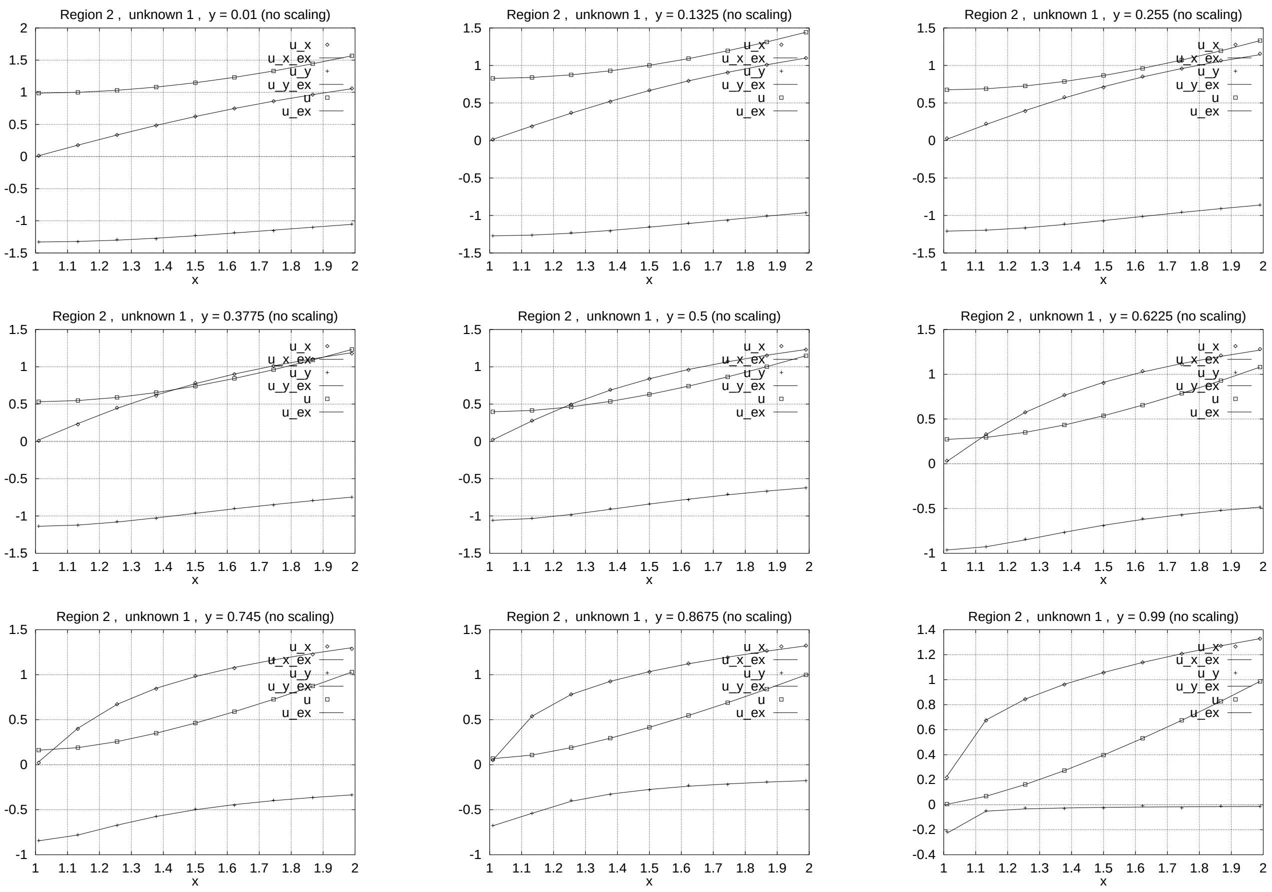


Figure 14: Comparison of exact and calculated u, u_x and u_y, region 2.

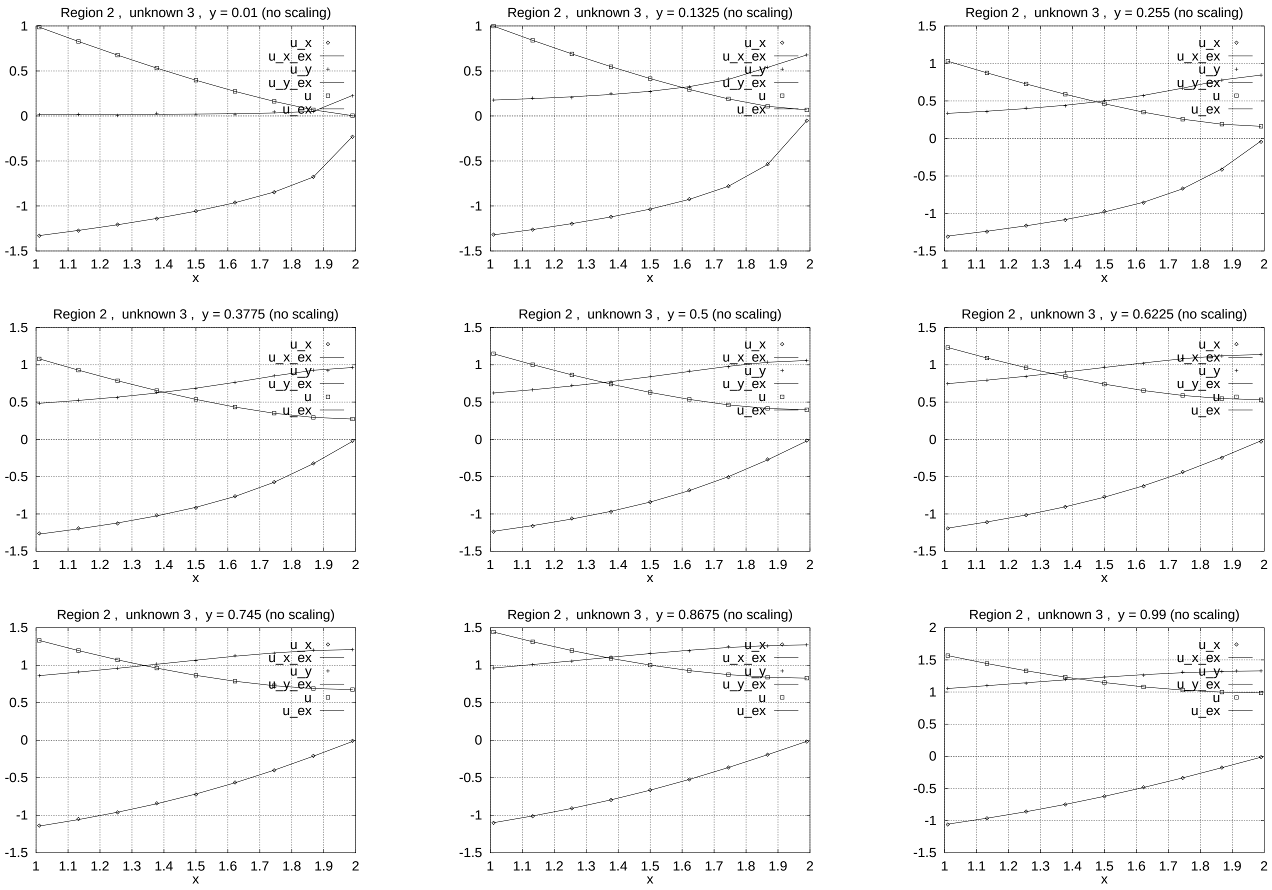


Figure 15: Comparison of exact and calculated v, v_x and v_y, region 2.

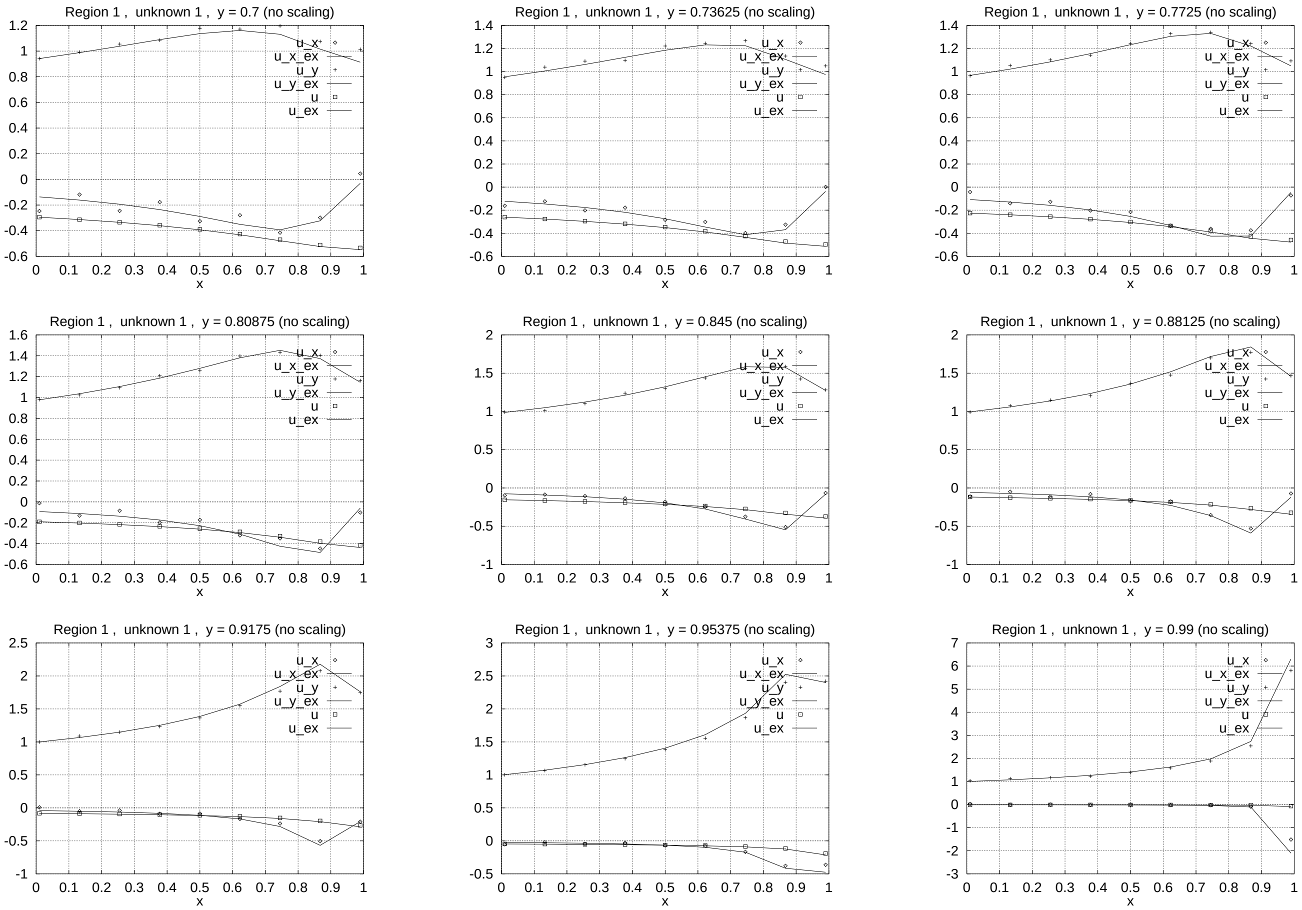


Figure 16: Comparison of exact and calculated u, u_x and u_y, region 1.

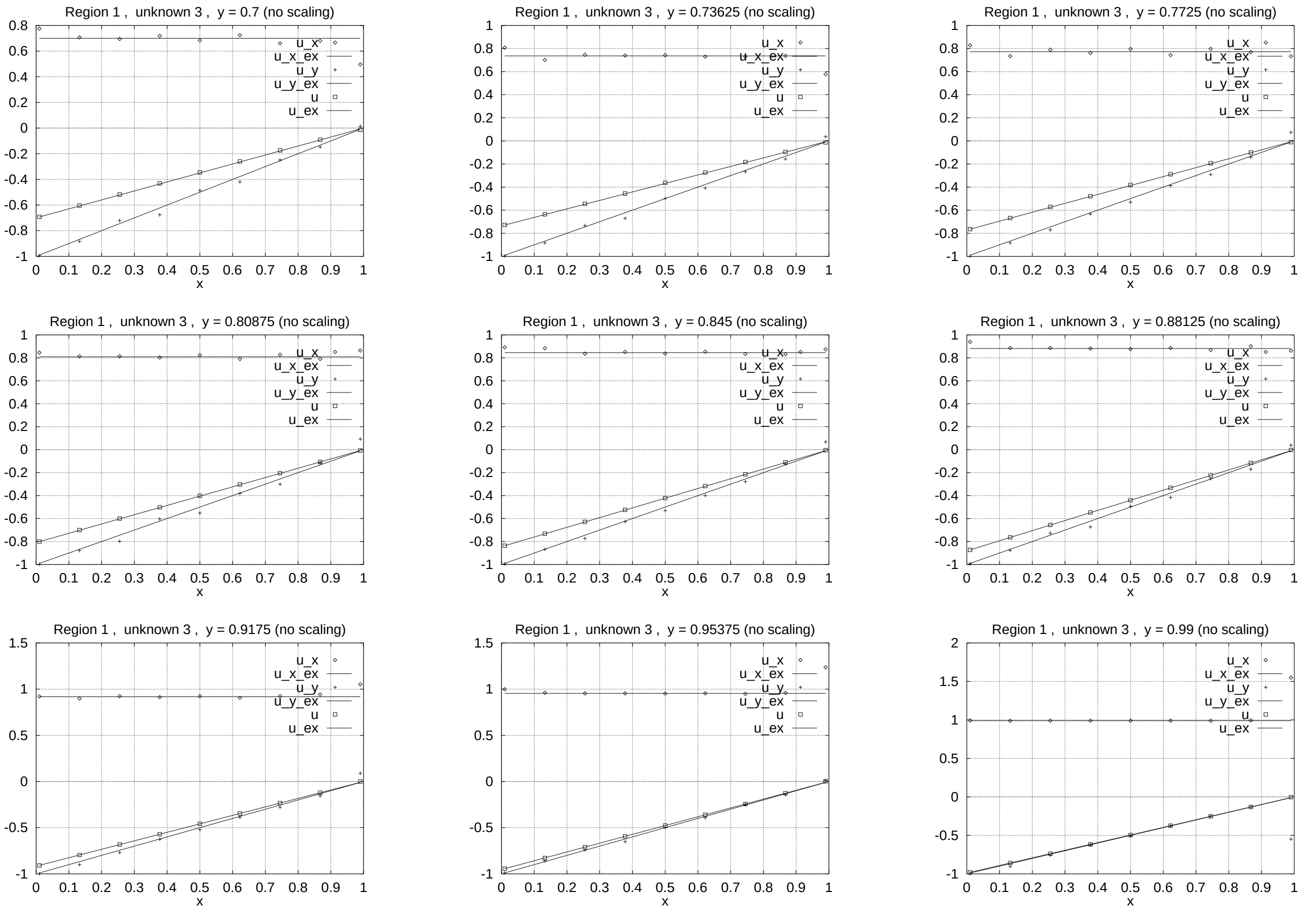


Figure 17: Comparison of exact and calculated v, v_x and v_y, region 1.

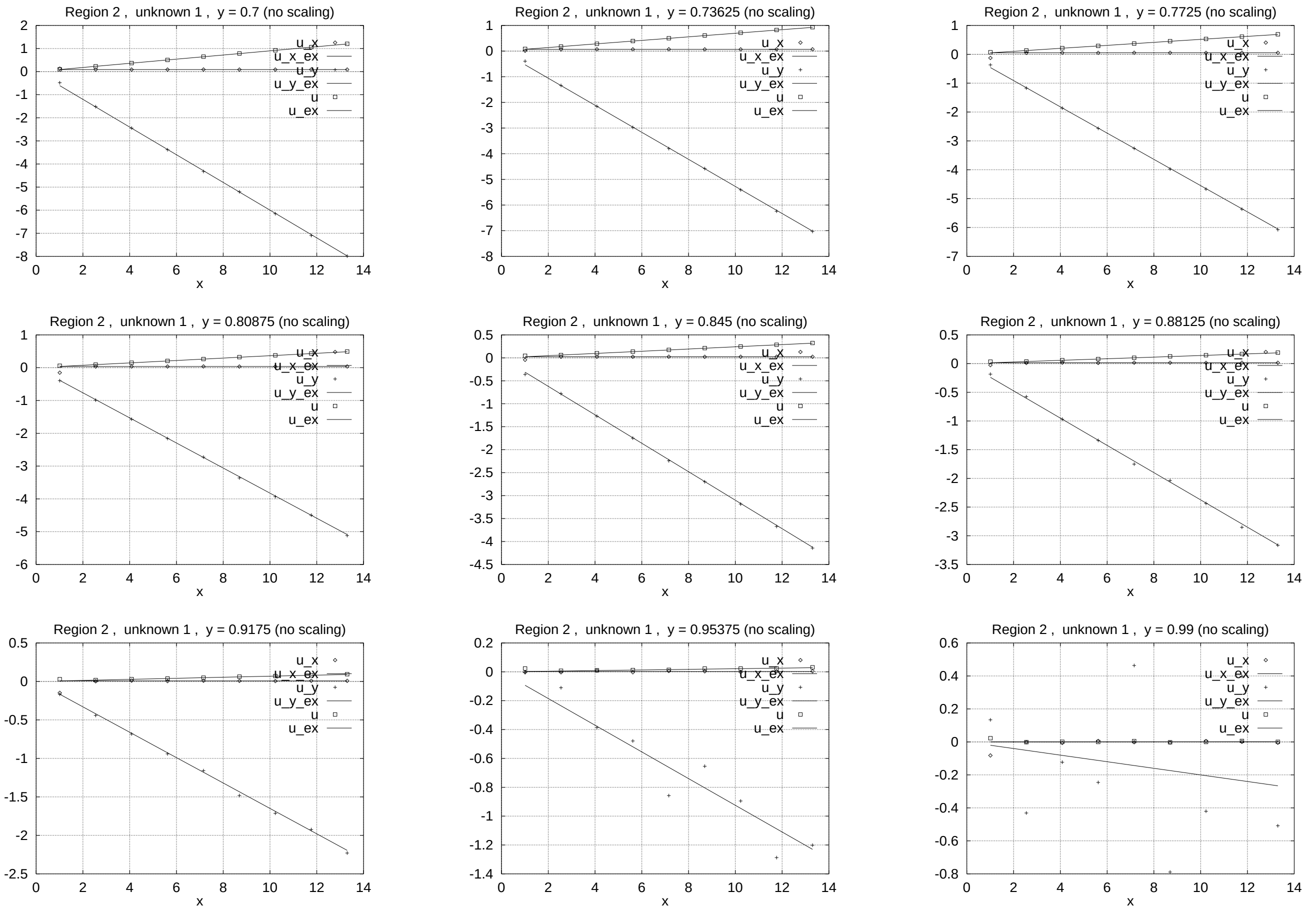


Figure 18: Comparison of exact and calculated u, u_x and u_y, region 2.

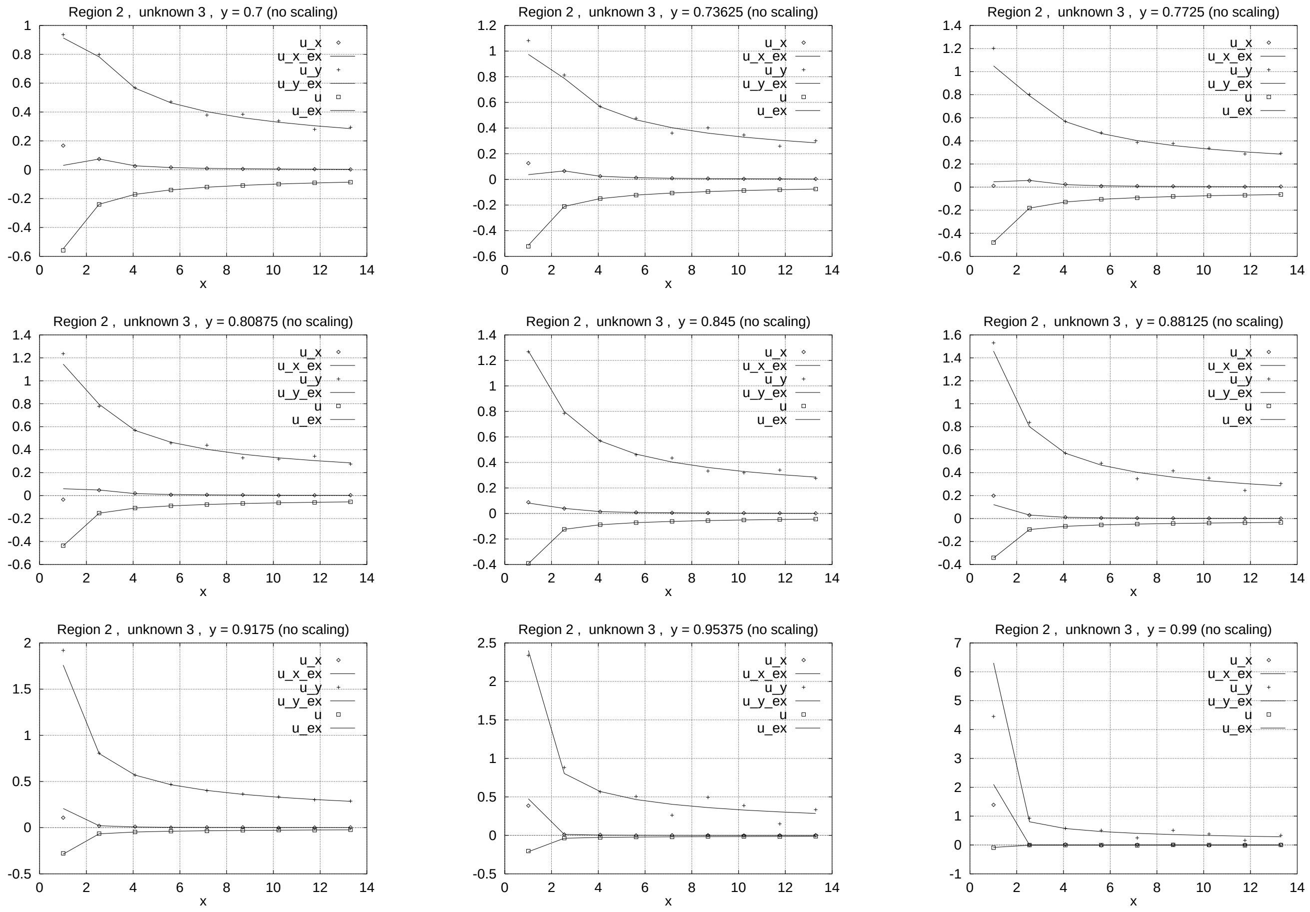


Figure 19: Comparison of exact and calculated v, v_x and v_y, region 2.