Moduli spaces of polygons in symmetric spaces

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1 Polygons in symmetric spaces

Let X be a symmetric space of noncompact type, then the group G of (orientation preserving, OR SHOULD WE TAKE THE IDENTITY COMPONENT OF THE ISOMETRY GROUP?) isometries of X acts on $X \times X$ diagonally so that the quotient X^2/G is naturally isomorphic to the Weyl chamber W of X. Let $\theta : X^2 \to W$ be the projection. Geometrically the map θ can be described as follows. Identify W with a Weyl chamber in a maximal flat $F \subset X$, let o be the vertex of W. Pick an isometry $g \in G$ so that $g(a) = o$. Then there is an element k of the maximal compact subgroup $K \subset G$ (which fixes o) that moves $g(b)$ to W. Thus $kg(b) = \theta(a, b)$. We will also identify $\theta(a, b)$ with a K-orbit in the Lie algebra $\mathfrak{t}$ of the maximal torus $T \subset G$ so that $F = To$. We will think of $\theta(a, b)$ as the "generalized length" (combined length and direction) of the geodesic segment $[ab]$ connecting a to b: the direction of $[ab]$ is a point in the intersection pW of the unit sphere with W.

Fix an n-tuple $w = (w_1, ..., w_n)$ of elements $w_1, ..., w_n \in W - \{o\}$ with the lengths $r_1, ..., r_n$. We define the space $\tilde{N}_w(X)$ of *open n-gon linkages* with the "generalized side-lengths" $w = (w_1, ..., w_n)$ as

$$\{(x_1, ..., x_{n+1}) \in X^{n+1} : \theta(x_i, x_{i+1}) = w_i, 1 \le i \le n\}$$

The space $\tilde{M}_w(X)$ of *closed* n-gon linkages in X with the "generalized side-lengths" $w = (w_1, ..., w_n)$ is

$$\{(x_1, ..., x_{n+1}) \in \tilde{N}_w(X) : x_{n+1} = x_1\}$$

These definitions becomes more geometric if X is a symmetric space of rank 1 (different from $\mathbb{R}$). In this case the Weyl chamber of X is isometric to $\mathbb{R}_+$, hence $w = (w_1, ..., w_n) \in (R_+)^n$ and $\tilde{N}_w(X), \tilde{M}_w(X)$ are described as

$$\tilde{N}_w(X) = \{(x_1, ..., x_{n+1}) \in X^{n+1} : d(v_i, v_{i+1}) = w_i, 1 \le i \le n\}$$

$$\tilde{M}_w(X) = \{(x_1, ..., x_{n+1}) \in \tilde{N}_w(X) : x_{n+1} = x_1\}$$

i.e. these are spaces of open (resp. closed) polygons in X with the side-lengths $(w_1, .., w_n)$. To get an alternative description of $\tilde{N}_r(X)$ take Γ be the metric graph with the vertices $v_1, ..., v_{n+1}$, edges $e_j = [v_j, v_{j+1}], j = 1, ..., n$, where each edge e_j has the length $w_j > 0$. Then $\tilde{N}_w(X)$ is canonically identified with the space $Is(\Gamma, X)$ of maps $\Gamma \to X$ which are isometric on the edges of Γ. If Γ' is the graph obtained from Γ by identifying x_1 and x_{n+1} then $\tilde{M}_w(X) \cong Is(\Gamma', X)$.

We will identify two polygons $P = (v_1, ..., v_n)$ and $Q = (u_1, ..., u_n)$ in X if and only if there exists an orientation preserving isometry g of X which sends the vertices of P to the vertices of Q, that is

$$gv_i = u_i, \qquad 1 \le i \le n$$

Equivalently, two maps $\phi, \psi \in Is(\Gamma, X)$ are identified if and only if there exists $g \in G = Isom_+(X)$ such that $\psi = g \circ \phi$. The quotients $N_w(X) = \tilde{N}_w/G$ and $M_w(X) = \tilde{M}_w/G$ are called the *moduli spaces* of open and closed polygonal linkages with the generalized side-lengths $(w_1, ..., w_n)$. In what follows we normalize w so that $|w_1| + ... + |w_n| = 1$.

The above definitions admit obvious generalizations to the spaces of curves in X. Namely, fix a (measurable?) mapping $\nu : I = [0,1] \to pW$, this mapping will prescribe the "direction" of the tangent vector field of the curve. Define $IMaps_\nu(I, X)$ as the collection of (Lipschitz??) maps $f : I \to X$ so that $|f'(s)| = 1$ and $p\theta f'(s) = \nu(s)$ for almost all $s \in I$. Let $w_i \in W$ be unit vectors, $i = 1, ..., n$. If ν is a piecewise-constant map so that $\nu([s_i, s_{i+1})) = pw_i$ where $s_{i+1} - s_i = r_i > 0$, $i = 1, ..., n$, then $IMaps_\nu(I, X) \cong \tilde{N}_w(X)$ with $w = (r_1 w_1, ..., r_n w_n)$.

We will also need the "intermediate" objects between $\tilde{N}_w$ and N_w, $\tilde{M}_w$ and M_w. Fix a point $* \in X$. Then let $\tilde{N}_w(X, *) = \{(x_1, ..., x_{n+1}) \in \tilde{N}_w(X) : x_1 = *\}$, $\tilde{M}_w(X, *) = \{(x_1, ..., x_{n+1}) \in \tilde{M}_w(X) : x_1 = *\}$. Then $N_w(X)$ and $M_w(X)$ are the quotients of $\tilde{N}_w(X, *)$ and $\tilde{M}_w(X, *)$ by the maximal compact subgroup $K \subset G$ which is the stabilizer of $*$.

Also define the spaces $Pol_n(X) = (x_1, ..., x_{n+1}) \in X^{n+1}$ of open n-gons and $Pol_n(X, *) = (*, x_2, .., x_{n+1})$ of the based open n-gons in X. Let $CPol_n(X), CPol_n(X, *)$ be the spaces of closed polygons and closed based polygons respectively.

2 Stable measures and stable maps

Let $S := \partial_\infty(X)$ be the sphere at infinity of the symmetric space X, recall that it is defined as the space of equivalence classes of complete geodesic rays ρ in X, where two rays are equivalent iff they are within finite distance from each other. The space S has natural topology and a natural smooth structure and is diffeomorphic to the sphere S^{N-1} where N is the dimension of X.

Let μ be a probability measure on S. Consider the Busemann functions $b_\lambda : X \to \mathbb{R}$ normalized at the base-point $* \in X$. Then the measure μ determines the "averaged" function $b_\mu = \int_{S,\mu} b_\lambda$ on X. The measure μ is called *stable* if the function b_μ has a unique minimum and *semi-stable* if b_μ is bounded from below, and *nice semi-stable* if it has minimum in X.

We will consider only those probability measures which are supported on a single G-orbit on S. Then the above definitions coincide with the classical definitions from the geometric invariant theory provided that μ is a measure supported on a finite number of points and the mass of each point is in $\mathbb{Q}_+$. If $rank(X) = 1$ then G acts transitively on S and we can describe stable/semi-stable/nice semi-stable measures as follows:

A probability measure μ on S is *stable* (*semi-stable*) if the mass of each point of S is less than $1/2$ (is at most $1/2$). A probability measure μ is *nice semi-stable* if it is either stable or is supported on a 2-point subset of S where each point has the mass $1/2$. The spaces of stable and nice semi-stable probability measures on S are denoted $Measures(S)_{st}$ and $Measures(S)_{nsst}$ respectively.

See [4] for the detailed discussion of stable measures. Note that a measure supported on the ideal boundary of a proper parallel set Par_l of a geodesic $l \subset X$ is unstable since the averaged Busemann function b_μ is invariant under translations along l and hence its minimum (if exists) cannot be unique.

Hereditary properties of stable measures. Let $X' \subset X$ be a totally-geodesic symmetric subspace in X. Let μ' be a measure on $S' = \partial_\infty X'$ and μ is its push-forward to $S = \partial_\infty X$. Then $mu' \in Measures(S)_{nsst} \Rightarrow mu' \in Measures(S)_{nsst}$. If the parallel set of X' in X equals X' then $mu' \in Measures(S)_{st} \Rightarrow mu' \in Measures(S)_{st}$.

Suppose that $X = Y \times E$ be the decomposition of X so that E is the maximal Euclidean factor. Then $\partial_\infty X = \partial_\infty Y \vee \partial_\infty E$, where $\vee$ is the metric suspension. Suppose that μ is a nice semi-stable measure on $\partial_\infty X - \partial_\infty E$, then the projection of μ to $\partial_\infty Y$ is also nice semi-stable ???

Let Σ_r be the finite set $\{1, ..., n\}$ with the probability measure r where the point i is given the mass r_i. We will consider the space of maps $Maps(\Sigma_r, S)$, which is given the product topology. Given $f \in Maps(\Sigma_r, S)$ we define the push-forward $f_*(r)$, which is a finite measure μ on S. This defines the push-forward map

$$\epsilon : Maps(\Sigma_r, S) \to Measures(S)$$

Then we define the collections of *stable maps* $Maps(\Sigma_r, S)_{st}$ and *nice semi-stable maps* $Maps(\Sigma_r, S)_{nsst}$ to be $\epsilon^{-1}(Measures(S)_{st}$ and $\epsilon^{-1}(Measures(S)_{nsst}$ respectively.

More generally, consider the space $Maps(I, S)$ of measurable maps for the unit interval I (with the Lebesgue measure) to S. Then we define the collections of *stable maps* $Maps(I, S)_{st}$ and *nice semi-stable maps* $Maps(I, S)_{nsst}$ to be $\epsilon^{-1}(Measures(S)_{st}$ and $\epsilon^{-1}(Measures(S)_{nsst}$ respectively, where ϵ is the push-forward of the Lebesue measure. To relate $Maps(I, S)_{st}$ and $Maps(\Sigma_r, S)_{st}$ note that given $r = (r_1, ..., r_n)$ we can use the subset of $Maps_r(I, S) \subset Maps(I, S)$ which consists of piecewise-constant maps $\nu : I \to S$ which are constant on the intervals $[s_i, s_{i+1})$, where $s_{i+1} - s_i = r_i$, $i = 1, ..., n$. Then $Maps(I, S)_{st} \cap Maps_r(I, S)$ is canonically isomorphic to $Maps(\Sigma_r, S)_{st}$ and the same is true for semi-stable and nice semi-stable maps.

Given a directed geodesic segment $e = [xy] \in X$ we will define the *infinite extension* of $[xy]$ to be the complete geodesic ray ρ_e in X that has x as its initial point and which contains e.

Suppose that $P \in \tilde{N}_w(X)$ is an open polygonal linkage, where $r_i = |w_i|$. Each edge $e_i = [v_i, v_{i+1}]$ of P determines the complete geodesic ray ρ_{e_i} which is the infinite extension of e_i. Thus the linkage determines a map $f : \Sigma_r \to S$ by $i \mapsto [\rho_i]$, where $[\rho_i] \in S$ is the equivalence class of the ray ρ_i. This gives us a map

$$\tilde{N}_w(X) \to Maps(\Sigma_r, S)$$

which will be called the *generalized Gauss map* $\hat{\gamma}$. Note that the map $\hat{\gamma}$ is $G = Isom_+(X)$-equivariant, thus it projects to a map $\gamma : N_w(X) \to Maps(\Sigma_r, S)/G$. Both $\hat{\gamma}$ and γ are clearly continuous mappings. To generalize these maps to the case of curves in X take $f \in IMaps_\nu(I, X)$, then $\hat{\gamma}(f)(t) = [\rho_t]$ where ρ_t is the complete geodesic ray in X with the initial velocity $f'(t)$, define for almost all $t \in I$.

Let V_λ be the gradient vector field of the function $-b_\lambda$ and $F_{\lambda,t} : X \to X$ be the integral flow of V_λ. Then $F = F_\lambda$ is isomorphic to the restriction of the geodesic flow of X to a stable submanifold. Thus for each $\xi \in T_x X$ we have

$$\|DF_t(\xi)\| \leq \|\xi\| \tag{1}$$

with the equality iff $t = 0$ or ξ tangent to the flat through x which is asymptotic to $\lambda \in \partial_\infty X$. Note that the vector field $\xi(t) = DF_t(\xi)$ is the Jacobi field along the geodesic $F_t(x), t \in \mathbb{R}$. Thus

$$\frac{d}{dt}\langle \xi(t), \xi(t) \rangle \leq 0 \tag{2}$$

with equality iff the the curvature vanishes on the span of $\xi(t)$ and $V_\lambda(x_t)$, $x_t = F_t(x)$, i.e. ξ is tangent to the flat through x which contains the geodesic x_t.

Remark 2.1 *If $rank(X) = 1$ then the equality in (2) holds iff ξ is tangent to the flow line x_t of F through x.*

Note that for each open polygonal linkage $P=[x_1,...,x_{n+1}]$ in X we have:

$$x_{i+1}=F_{\lambda_i,r_i}(x_i)$$

where λ_i is the equivalence class of the ray ρ_{e_i}. Hence

$$x_{n+1}=F_{\lambda_n,r_n}\circ ...\circ F_{\lambda_1,r_1}(x_1)$$

We now fix $\psi\in Maps(\Sigma_r,S)_{nsst}$ so that $\psi(i)=\lambda_i$. This yields the map

$$\Phi_\psi : X\to X, x\mapsto F_{\lambda_n,r_n}\circ ...\circ F_{\lambda_1,r_1}(x)$$

This map is a composition of smooth 1-Lipschitz maps, therefore it is 1-Lipschitz itself. Note that P is a closed polygon iff x_1 is the fixed point of the map Φ_ψ.

Lemma 2.2 *Suppose that $\xi\in T_xX$ is a nonzero vector so that $\|D\Phi_\psi(\xi)\|=\|\xi\|$. Then $\psi(\Sigma_r)$ lies on the boundary of a parallel set M_σ of a geodesic σ in X. In particular, if $rank(X)=1$ then $\|D\Phi_\psi(\xi)\|=\|\xi\|$ implies that the polygon P is degenerate and ξ is tangent to P.*

Proof: Let $\xi_1=\xi, \xi_{i+1}:=DF_{\lambda_i,r_i}\xi_i, i=1,...,n$ be the collection of vectors in $T_{x_i}X$. Then the equality $\|\xi_{n+1}|=\|D\Phi_\psi(\xi)\|=\|\xi\|$ implies that for each i the vector ξ_{i+1} is obtained from ξ_i by parallel transport along the geodesic $[x_i,x_{i+1}]$ and both vectors ξ_{i+1}, ξ_i are tangent to a flat Π_i in X through $[x_i,x_{i+1}]$. In particular, the geodesics $\sigma_i=\exp(\mathbb{R}\xi_i), \sigma_{i+1}=\exp(\mathbb{R}\xi_{i+1})$ are parallel in X for each i. Therefore all the geodesic σ_i, $i=1,...,n$ are mutually parallel. Let M_σ denote the parallel set of $\sigma=\sigma_1$, then $\sigma_i\subset M_\sigma$, $\Pi_i\subset M_\sigma$ for each i. However the flat Π_i is asymptotic to λ_i. The lemma follows. $\Box$

Corollary 2.3 *Suppose that ψ is such that $\gamma^{-1}(\psi)\subset M_w(X)$ contains more than one polygon. Then ψ is unstable.*

Proof: If ψ is stable then $\psi(\Sigma_r)$ does not belong to the ideal boundary of any parallel set in X, see **??**. Hence the associated map $\Phi_\psi : X\to X$ is a contraction, which implies that it has at most one fixed point x_1. The polygon $P\in\gamma^{-1}(\psi)$ is completely determined by the initial point x_1, hence $\gamma^{-1}(\psi)\subset M_w(X)$ contains of at most one point. $\Box$

We now analyse the the nice semi-stable case. ????

Generalization to the case of C^1-smooth (??) curves in X. We start with a (continuous ?) function $\lambda:[0,1]\to\partial_\infty X$. This determines a continuous family $V(x,t)$ of vector Jacobi fields on X where $V(x,t)=V_{\lambda,t}=-\nabla b_{\lambda(t)}$. Consider the nonautonomous 1-st order ODE on X:

$$\frac{dx}{dt}=V(x,t) \tag{3}$$

Let $x(s,t)$ be a 1-parameter family of solutions of the equation (3), $x(t):=x(t,0)$, with the variational vector field $\xi(t)=\frac{\partial x}{\partial s}|_{s=0}$. Then ξ satisfies the equation

$$\frac{D}{dt}\xi(t)=\frac{D}{dt}\frac{\partial x}{\partial s}|_{s=0}=\frac{D}{ds}\frac{\partial x}{\partial t}|_{s=0}=\frac{D}{ds}V(x,t)|_{s=0}=(gradV)\cdot(\xi) \tag{4}$$

where $\frac{D}{dt}$ is the covariant derivative along $x(t)$,

$$gradV=(\nabla_{\frac{\partial}{\partial x_1}}V,\dots,\nabla_{\frac{\partial}{\partial x_n}},(\xi)=(\xi_1,...,\xi_n)$$

$\xi = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i}$ and

$$(gradV) \cdot (\xi) = \sum_{i=1}^n \xi_i \nabla_{\frac{\partial}{\partial x_i}} V$$

Suppose for a moment that the tangent vector field along the curve $x(t)$ satisfies the variational equation, then

$$\frac{D}{dt}\frac{\partial x}{dt} = (\nabla V) \cdot (\frac{\partial x}{dt})$$

$$\frac{D}{dt}\frac{\partial}{dt} x(t) = \frac{D}{dt} F(x(t), t) = (\nabla V) \cdot (\frac{D}{dt} x(t)) + \frac{\partial F}{dt}$$

Thus $\frac{\partial F}{dt} \equiv 0$ for each x on the curve $x(t), t \in [0, 1]$. However if $\lambda(t)$ is not constant then $F(x, t)$ is nonconstant as the function of t for any given $x \in X$. We conclude that $\xi = \frac{\partial x}{dt}$ never satisfies the variational equation (4) unless $\lambda(t)$ is constant.

Consider now the behavior of $\|\xi(t)\|$ where ξ satisfies the variational equation (4).

$$\frac{d}{dt}\langle \xi(t), \xi(t) \rangle = 2\langle \xi(t), \frac{D}{dt}\xi \rangle = 2\langle \xi(t), (gradV(t)) \cdot (\xi(t)) \rangle, t = t_0$$

We replace $V(t)$ by the time-independent vector field $V(t_0)$, then the variational vector field ξ_0 along the geodesic $x_0(t)$ tangent to $V(t_0)$ satisfies the same equation

$$\frac{d}{dt}\langle \xi_0(t), \xi_0(t) \rangle = 2\langle \xi_0(t), (gradV(t_0)) \cdot (\xi(t)) \rangle, t = t_0 \quad \text{and} \quad \xi_0(t_0) = \xi(t_0)$$

Thus $\langle \xi_0(t_0), (gradV(t_0)) \cdot (\xi(t_0)) \rangle \leq 0$ with equality if and only if $\xi(t_0)$ is a multiple of $V(t_0)$. We conclude that the length of $\xi(t)$ is not increasing and $\|\xi(0)\| > \|\xi(1)\|$ unless for each $t \in [0, 1]$ the vector field $\xi(t)$ is a multiple of $V(t)$. However $\|V(t)\| \equiv 1$, hence the only case when $\|\xi(0)\| = \|\xi(1)\|$ is $\xi(t) = \pm V(t)$ which implies that $\lambda(t)$ is a constant function. This proves

Lemma 2.4 *Suppose that $rank(X) = 1$ and $\lambda(t)$ is continuous. Then the equation (3) has at most one periodic solution $x = x(t)$.*

We now prove the existence theorem for periodic solutions of the equation (3) under the assumption that $\lambda(t)$ is stable. Note that the map $F : X \to X$ given by $x(0) \mapsto x(1)$ where $x(t)$ satisfies the equation (3) is a strict construction on compacts in X in the cases

(a) $\lambda(t)$ is piecewise-constant.

(b) $\lambda(t)$ is continuous.

Theorem 2.5 *Suppose that $\lambda : I \to S$ is a stable measurable map so that the associated mapping $F : X \to X$ is (locally) a uniform contraction. Then*

(i) The mapping F has a (unique) fixed point $x_* \in X$.

(ii) The solution of the initial value problem

$$x(0) = x_*, x'(t) = V(x, t)$$

is periodic, i.e. in the polygonal case we get a closed polygon P with the given Gauss map and the the case of curves we get a smooth closed curve $x(t)$ with satisfying the equation (3).

Proof: Let H be a closed convex subset in X with smooth boundary. Consider the function $\tau = \tau_H : X \to \mathbb{R}$ defined as $\tau(x) = d(x, H)$. It is known that the function τ is smooth 1-Lipschitz, it has the slope -1 along the geodesics in $X - H$ orthogonal to ∂H (which are the gradient lines of τ) parameterized towards H. Let $v \in T_x(X) - \{0\}$, α is the angle which v makes with the vector $-grad(\tau)$. Then $\frac{\partial \tau}{\partial v} = -|v| \cos(\alpha)$.

Lemma 2.6 *Let $x \in X - Nbd_r(H)$, $y, z \in H$. Then the angle α between $[yx]$ and $[zx]$ satisfies*

$$1 - \cos(\alpha) \leq \frac{\delta^2}{(r-\delta)^2}$$

where δ is the hyperbolicity constant of X.

Proof: The triangle $[xyz] \subset X$ is δ-thin, thus there exist points $p \in [xy], q \in [zx]$ and $w \in [yz] \subset H$ which are within the distance $\leq \delta$ from each other. Thus $d(q,x) \geq r - \delta$, $d(p,x) \geq r - \delta$ and $d(p,q) \leq \delta$. Now we use the comparizon of the triangle $[xpq]$ with the Euclidean triangle with the same side-lengths and the cosine formula to conclude that

$$1 - \cos(\alpha) \leq \frac{\delta^2}{(r-\delta)^2} \qquad \square$$

From now on we normalize the metric on X so that $\delta = 1$.

Let $\lambda : [0,1] \to \partial_\infty X$ be a measurable map, μ is the measure on $\partial_\infty X$ which is the push-forward measure of the Lebesgue measure on $[0,1]$ via λ. Suppose that $x(t), t \in [0,1] = I$ is a solution curve in X for the differential equation (3). Let $Q \subset \partial_\infty X$ be a closed subset such that $\mu(Q) \geq m$. Assume that the image of the curve $x(t)$ is disjoint from the closed r-neighborhood of the convex hull H of Q in X, where $r \geq 2$. Then we have

Lemma 2.7 *Under the above conditions*

$$\tau_H(x(1)) - \tau_H(x(0) \leq 1 - 2m + \frac{m}{(r-1)^2} \quad .$$

Proof: Let $\tau(t) := \tau_H(x(t))$, this function is differentiable a.e. and has measurable derivative. We have

$$\tau(1) - \tau(0) = \int_0^1 \frac{d\tau}{dt} dt = \int_{I_1} \frac{d\tau}{dt} dt + \int_{I_2} \frac{d\tau}{dt} dt$$

where $I_1 = \{t : x'(t) = V_{\lambda(t)}(x(t)), \lambda(t) \in Q\}, I_2 = I - I_1$. Since $\|x'(t)\| \equiv 1$ and τ is 1-Lipshitz,

$$\int_{I_2} \frac{d\tau}{dt} dt \leq \int_{I_2} dt = mes(I_2) \leq 1 - m$$

On the other hand, if $t \in I_1$ and α_t is the angle between $x'(t)$ and the shortest geodesic from $x(t)$ to H, then (according to Lemma 2.6)

$$1 - \cos(\alpha_t) \leq \frac{1}{(r-1)^2}$$

Hence for $t \in I_1$ we have

$$\frac{d\tau}{dt} = -\cos(\alpha_t) \leq \frac{1}{(r-1)^2} - 1$$

which implies

$$\int_{I_1} \frac{d\tau}{dt} dt \leq mes(I_1)(\frac{1}{(r-1)^2} - 1) \leq m(\frac{1}{(r-1)^2} - 1)$$

$$\tau(1) - \tau(0) \leq 1 - m + m(\frac{1}{(r-1)^2} - 1) = 1 - 2m + \frac{m}{(r-1)^2} \qquad \square$$

Let $m > 1/2$, then there is $r(m)$ such that

$$1 - 2m + \frac{m}{(r(m)-1)^2} < 0$$

Let

$$r_* := \max(r(m), 2) \quad . \tag{5}$$

Corollary 2.8 *Suppose that $m > 1/2$, then for any $x \in Nbd_{r_*}(H)$ the orbit $F^k(x), k \in \mathbb{Z}^+$ is contained in $Nbd_{r_*+2}(H)$.*

Proof: If $x_j = F^j(x) \in Nbd_{r_*+1}(H)$ then $F(x_j) \in Nbd_{r_*+2}(H)$. Suppose that $x_j \in Nbd_{r_*+2}(H) - Nbd_{r_*+1}(H)$. Then the curve $x(t)$ with the initial value $x(0) = x_j$ satisfying the equation (3) is disjoint from $Nbd_{r_*}(H)$ (since its length equals 1) and hence by applying the previous lemma we conclude that

$$\tau_H(F(x_j)) - \tau_H(x_j) < 0$$

which means that $F(x_j) \in Nbd_{r_*+2}(H)$. $\square$

We now use the assumption that μ is a a stable probability measure on $S = \partial_\infty X$. Then there is $m > 1/2$ such that for each point $s \in S$ there is a compact subset $Q_s \subset S - \{\lambda\}$ so that $\mu(Q_s) \geq m$ (this follows from the fact that at most two points s_1, s_2 in S have mass more than 0.4, then let $m = \min(1 - \mu(s_1), 0.5 - \mu(s_2))$. The collection of open sets $\{S - Q_s, s \in S\}$ forms a covering of S, hence we may choose a finite subcovering $\{S - Q_{s_j}, j = 1, ..., q\}$. It follows that

$$\cap_{j=1}^q Q_{s_j} = \emptyset$$

Let H_j denote the closed convex hull to Q_{s_j} in X. Recall that according to [1] the closure of H_j in $X \cup \partial_\infty X$ equals Q_{s_j}, hence the same holds for the closure of its tubular neighborhood $Nbd_r H_j$.

Lemma 2.9 *For each $r \geq 0$ the intersection*

$$C_r = \cap_{j=1}^q Nbd_r H_j \tag{6}$$

is a bounded subset of X.

Proof: The intersection C is convex. If it is unbounded then C contains a complete geodesic ray ρ asymptotic to a point $[\rho] \in S$. Since $\rho \subset Nbd_r H_j$ for each j we conclude that $[\rho] \in \partial_\infty Nbd_r H_j = Q_{s_j}$, which means that the sets Q_{s_j} have nonempty intersection. This contradicts our choice of the sets Q_{s_j}. $\square$

The next lemma follows directly from Corollary 2.8.

Lemma 2.10 *Let $r = r_*$ where r_* is given by equation (5). Define $C := C_{r_*}$ and $C^+ := C_{r_*+2}$ using the equation (6). Then for each $x \in C$ the orbit $F^k(x), k \in \mathbb{Z}^+$ is contained in C^+.*

Finaly we can prove Theorem 2.5. The set C^+ is bounded and the mapping F is a uniform contraction on C^+. Hence for each $x \in C$ the sequence $x_k := F^k(x)$ is convergent in X to a fixed point of the mapping $F : X \to X$. $\square$

Theorem 2.11 *The mapping $\gamma : M_w \to Maps(\Sigma_r, S)/G$ is injective provided that $rank(X) = 1$.*

Proof:

Theorem 2.12 *Suppose that $rank(X) = 1$. Then the image of the map $\gamma : M_w(X) \to Maps(\Sigma_r, S)/G$ equals the set of nice semi-stable maps $Maps(\Sigma_r, S)_{nsst}/G$. The projection γ is a homeomorphism.*

Lemma 2.13 *The image of $\gamma : M_w(X) \to Maps(\Sigma_r, S)/G$ is contained in $Maps(\Sigma_r, S)_{nsst}/G$.*

Proof: Recall that each point $\lambda \in S$ determines a Busemann function b_λ, defined up to an additive constant. The level sets of b_λ are called *horospheres* centered at λ. The Busemann function is a linear function with the slope -1 on each geodesic σ asymptotic to λ (the geodesic σ is parameterized so that $\lim_{t\to\infty} \sigma(t) = \lambda$. We will identify the function $b_\lambda : X \to \mathbb{R}$ with the projection of X to σ along the horospheres centered at λ. Thus b_λ is 1-Lipschitz and $|Db(v)| < |v|$ for each $v \in TX$ unless v is tangent to a geodesic asymptotic to λ.

Suppose that P is a polygon in X so that $\gamma(P)$ is unstable, then there is a point $\lambda \in S$ so that $r_I := r_{i_1} + ... + r_{i_k} \geq 1/2$ where $I = \{e_{i_j}, 1 \leq j \leq k\}$ is the subset of the edges of P whose extensions ρ_{e_i} are asymptotic to λ. Let $b = b_\lambda$ be a Busemann function associated to λ. Then $|b(x_{i+1}) - b(x_i)| \leq r_i$, and the equality holds if and only if $\rho_{[x_i,x_{i+1}]} = \lambda$ or $\rho_{[x_{i+1},x_i]} = \lambda$. Therefore

$$0 = \sum_{i\in\mathbb{Z}/n} b(x_{i+1}) - b(x_i) = r_I + t$$

where $|t| \leq \sum_{e_i \notin I} |e_i| = r_{\bar{I}}$ and the equality holds iff each edge e of P which is not in I lies on a geodesic asymptotic to λ. Thus $r_I = -t$ However $r_{\bar{I}} < r_I$ if $\gamma(P)$ is not semi-stable and we get a contradiction in this case. If $r_{\bar{I}} = r_I$ then $r_I > |t|$ unless each edge of P lies on a geodesic asymptotic to λ. In the latter case the polygon P is degenerate and its image under $\tilde{\gamma}$ is nice semi-stable. $\square$

Let V_λ be the gradient vector field of the function $-b_\lambda$ and $F_{\lambda,t} : X \to X$ be the integral flow of V_λ. Then $F = F_\lambda$ is isomorphic to the restriction of the geodesic flow of X to a stable submanifold. Thus for each $\xi \in TX$ we have

$$\|DF_t(\xi)\| \leq \|\xi\|$$

with the equality iff $t = 0$ or ξ tangent to a flow line of F. Note that for each open polygonal linkage $P = [x_1, ..., x_{n+1}]$ in X we have:

$$x_{i+1} = F_{\lambda_i,r_i}(x_i)$$

where λ_i is the equivalence class of the ray ρ_{e_i}. Hence

$$x_{n+1} = F_{\lambda_n,r_n} \circ ... \circ F_{\lambda_1,r_1}(x_1)$$

We now fix $\psi \in Maps(\Sigma_r, S)_{nsst}$ so that $\psi(i) = \lambda_i$. This yields the map

$$\Phi_\psi : X \to X, x \mapsto F_{\lambda_n,r_n} \circ ... \circ F_{\lambda_1,r_1}(x)$$

This map is a composition of smooth 1-Lipschitz maps, therefore it is 1-Lipschitz itself. If $\xi \in TX$ is such that $\|D\Phi_\psi(\xi)\| = \|\xi\|$ then ξ is tangent to $e_1 = [x_1, x_2]$, $DF_{\lambda_1,r_r}(\xi)$ is tangent to both e_1 and e_2, etc. Thus in this case the polygon $P = [x_1, ..., x_{n+1}]$ is degenerate and ψ is not stable. We conclude that for a stable ψ the map $\Phi_\psi : X \to X$ is a contraction at each point $x \in X$ (but not necessarily a uniform contraction). This proves

Proposition 2.14 *Assume that ψ is stable. Then the mapping Φ_ψ has at most one fixed point in X.*

Note however that the polygonal linkage $[x_1, x_2, ..., x_{n+1}], x_{i+1} = F_{\lambda_i,r_i}(x_i)$ is closed if and only if $x_{n+1} = x_1$, i.e. x_1 is a fixed point of Φ_ψ. Thus the restriction of the map $\tilde{\gamma}$ to $\gamma^{-1}(Maps(\Sigma_r, S)_{st})$ is injective. It is clear that $\gamma^{-1}(Maps(\Sigma_r, S)_{nsst} - Maps(\Sigma_r, S)_{st})$ consists of degenerate polygons only. Two such polygons are obtained from each other by a translation along the geodesic connecting the two points on which the corresponding semi-stable measure is supported. Hence

Corollary 2.15 *The mapping* $\gamma : M_r(X) \to Maps(\Sigma_r, S)_{nsst}/Isom_+(X)$ *is injective.*

3 Generalization to the case of curves

Proposition 3.1 *Let* $rank(X) = 1$. *Consider* $C^0Maps(I,S)$ *and* $C^1IMaps(S^1,X)$ *the subspaces of continuous and continuously differentiable maps in* $Maps(I,S)$ *and* $IMaps(S^1,X)$. *Then the mapping* $\gamma : C^1IMaps(S^1,X)/G \to C^0Maps(I,S)_{nsst}/G$ *is a bijection.*

4 The Poisson-Lie group structure on B^n

Let G be a (linear) complex simple group, B be a Borel subgroup and K be a maximal compact subgroup. We will construct a Poisson structure on G which will restrict to a Poisson structure on B. Let R_g and L_g be the action of g on G by the right and left multiplication respectively.

Let $\mathfrak{g}$ denote the Lie algebra of G, $\mathfrak{k}$ be the Lie algebra of K and $\mathfrak{b}$ be the Lie algebra of B. Then $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{k}$ and $G = BK$. Let $\rho_{\mathfrak{k}}$ (resp. $\rho_{\mathfrak{b}}$) be the projection on $\mathfrak{k}$ (resp. on $\mathfrak{b}$). We define $\mathcal{R} := \rho_{\mathfrak{k}} - \rho_{\mathfrak{b}}$ and let $\langle , \rangle$ be the imaginary part of the Killing on $\mathfrak{g}$. In the direct sum splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{b}$ we see that $\mathfrak{k}$ and $\mathfrak{b}$ are totally-isotropic subspaces dually paired by $\langle , \rangle$.

Let $\varphi \in C^\infty(G)$. Define $D\varphi : G \to \mathfrak{g}$ and $D'\varphi : G \to \mathfrak{g}$ by

$$\langle D'\varphi(g), \nu \rangle = \frac{d}{dt}|_{t=0}\varphi(ge^{t\nu})$$

$$\langle D\varphi(g), \nu \rangle = \frac{d}{dt}|_{t=0}\varphi(e^{t\nu}g)$$

for $\nu \in \mathfrak{g}$.

We extend $\langle , \rangle$ to a biinvariant element of $C^\infty(G, S^2T^*(G))$ again denoted $\langle , \rangle$. Now define $\nabla\varphi \in C^\infty(G, T^*(G))$ by

$$\langle \nabla\varphi(g), x \rangle = d\varphi_g(x), x \in T_g(G)$$

We have

$$D\varphi(g) = dR_g^{-1}\nabla\varphi(g)$$

$$D'\varphi(g) = dL_g^{-1}\nabla\varphi(g) = Ad_{g^{-1}}D\varphi(g)$$

Define $\{\varphi, \psi\}$ for $\varphi, \psi \in C^\infty(G)$ by

$$\{\varphi, \psi\}(g) = \frac{1}{2}\langle \mathcal{R}D'\varphi(g), D'\psi(g)\rangle - \langle \mathcal{R}D\varphi(g), D\psi(g)\rangle$$

where $\mathcal{R} = \rho_{\mathfrak{k}} - \rho_{\mathfrak{b}}$.

Lemma 4.1 *The bracket* $\{\varphi, \psi\}$ *is a Poisson bracket on* $C^\infty(G)$.

Proof: See Theorem 1 of [8].

Let $w \in C^\infty(G, \Lambda^2T(G))$ be the bivector field corresponding to $\{\cdot, \cdot\}$. We now show that $\{\cdot, \cdot\}$ induces a Poisson bracket on $C^\infty(B)$.

Lemma 4.2 $w(b)$ *is tangent to* B , *i.e.* $w(b) \in \Lambda^2T_b(B) \subset \Lambda^2T(G)$ *for all* $b \in B$.

Proof: It suffices to prove that if φ vanishes identically on B then $\{\varphi,\psi\}$ vanishes identically on B for each ψ. However if φ vanishes identically on B then $\nabla\varphi(b)\in T_bB$ for all $b\in B$. Hence $D\varphi(b)\in\mathfrak{b}$, $D'\varphi\in\mathfrak{b}$ for each $b\in B$. This implies that $\mathcal{R}D\varphi=-D\varphi$, $\mathcal{R}D'\varphi=-D'\varphi$ and

$$2\{\varphi,\psi\}(b)=-\langle D'\varphi(b),D'\psi(b)\rangle+\langle D\varphi(b),D\psi(b)\rangle$$

But $D'\varphi(b)=Ad_{b^{-1}}D\varphi(b)$, $D'\psi(b)=Ad_{b^{-1}}D\psi(b)$ and $\langle,\rangle$ is Ad-invariant. $\square$

For the next corollary note that $T^*_b(B)$ is a quotient of $T^*_b(G)$.

Corollary 4.3 *Let π be the skew-symmetric 2-tensor on $T^*(G)$ corresponding to w. Pick $b\in B$ and $\alpha,\beta\in T^*_b(G)$. Then $\pi|b$ depends only on the images of α and β in $T^*_b(B)$.*

We will continue to use π for the skew-symmetric 2-tensor on $T^*(B)$ induced by π above.

Remark 4.4 *An argument identical to that above proves that $w(k)$ is tangent to K. Hence $\{\cdot,\cdot\}$ induces a Poisson structure on K. With the above structures K and B are sub Poisson Lie subgroups of the Poisson Lie group G.*

We will need a formula for the Poisson tensor π on B. We will use $\langle,\rangle$ to identify $T^*(G)$ and $T(G)$. Under this identification $T^*_b(B)$ is identified to $T_b(G)/T_b(B)$. We will identify this quotient with $dR_b\mathfrak{k}$. We let $\check{\pi}|_b$ denote the resulting skew-symmetric 2-tensor on $dR_b\mathfrak{k}$. Finally we define $\check{\pi}^r\in C^\infty(B,(\Lambda^2\mathfrak{k})^*)$ by

$$\check{\pi}|^r_b(x,y)=\check{\pi}|_b(dR_bx,dR_by),\quad x,y\in\mathfrak{k}$$

We now recover formulae (2.25) of [3] (or [6, Definition 4.2]) for $\check{\pi}^r$.

Lemma 4.5 $\check{\pi}|^r_b(x,y)=\langle\rho_{\mathfrak{k}}(Ad_{b^{-1}}x),\rho_{\mathfrak{b}}(Ad_{b^{-1}}y)\rangle$.

Proof: Choose $\varphi,\psi\in C^\infty(G)$ with $\nabla\varphi(b)=dR_bx$ and $\nabla\psi(b)=dR_by$. Then

$$\check{\pi}|^r_b(x,y)=\check{\pi}|_b(dR_bx,dR_by)=\check{\pi}|_b(\nabla\varphi(b),\nabla\psi(b))=$$

$$\pi|_b(d\varphi(b),d\psi(b))=\{\varphi,\psi\}(b)=\frac{1}{2}\langle\mathcal{R}dL_b^{-1}\nabla\varphi(b),dL_b^{-1}\nabla\psi(b)\rangle-$$

$$\frac{1}{2}\langle\mathcal{R}dR_b^{-1}\nabla\varphi(b),dR_b^{-1}\nabla\psi(b)\rangle=\frac{1}{2}\langle\mathcal{R}Ad_{b^{-1}}x,Ad_{b^{-1}}y\rangle-\frac{1}{2}\langle\mathcal{R}x,y\rangle$$

But $x,y\in\mathfrak{k}$ implies $\langle\mathcal{R}x,y\rangle=0$. Hence

$$\check{\pi}|^r_b(x,y)=\frac{1}{2}\langle\rho_{\mathfrak{k}}(Ad_{b^{-1}}x),Ad_{b^{-1}}y\rangle-\frac{1}{2}\langle\rho_{\mathfrak{b}}(Ad_{b^{-1}}x),Ad_{b^{-1}}y\rangle=$$

$$\frac{1}{2}\langle\rho_{\mathfrak{k}}(Ad_{b^{-1}}x),\rho_{\mathfrak{b}}Ad_{b^{-1}}y\rangle-\frac{1}{2}\langle\rho_{\mathfrak{b}}(Ad_{b^{-1}}x),\rho_{\mathfrak{k}}Ad_{b^{-1}}y\rangle=\langle\rho_{\mathfrak{k}}(Ad_{b^{-1}}x),\rho_{\mathfrak{b}}(Ad_{b^{-1}}y)\rangle$$

The last equality holds by skew-symmetry, see [6, Lemma 4.3]. $\square$

We will abuse notation and drop the ˇ and r in the notation for $\check{\pi}^r$ henceforth.

We now give G^n the product Poisson structure, hence B^n inherits the product structure. We introduce more notation to deal with the product. We let $\mathfrak{g}_i\subset\mathfrak{g}^n=\mathfrak{g}\oplus...\oplus g$ be the image of $\mathfrak{g}$ under the embedding into i-th summand. For $\varphi\in C^\infty(G^n)$ we define

$$D_i\varphi:C^\infty(G^n)\to\mathfrak{g}_i,\quad D'_i\varphi:C^\infty(G^n)\to\mathfrak{g}_i$$

as follows. Let $g=(g_1,...,g_n)\in G^n$ and $\nu\in\mathfrak{g}_i$, then

$$\langle D_i\varphi,\nu\rangle=\frac{d}{dt}|_{t=0}\varphi(g_1,...,e^{t\nu}g_i,...,g_n)$$

$$\langle D_i'\varphi, \nu\rangle = \frac{d}{dt}|_{t=0}\varphi(g_1, ..., g_i e^{t\nu}, ..., g_n)$$

Here we extend $\langle , \rangle$ to $\mathfrak{g}^n$ by

$$\langle \delta, \gamma\rangle = \sum_{i=1}^{n}\langle \delta_i, \gamma_i\rangle$$

for $\delta = (\delta_1, ..., \delta_n), \gamma = (\gamma_1, ..., \gamma_n)$. We define d_i, ∇_i in an analogous fashion. Finally define the Poisson bracket on $C^\infty(G^n)$ by

$$\{\varphi, \psi\}(g) = \frac{1}{2}\sum_{i=1}^{n}[\langle \mathcal{R}D_i'\varphi(g), D_i'\psi(g)\rangle - \langle \mathcal{R}D_i\varphi(g), D_i\psi(g)\rangle \quad .$$

As expected we obtain an induced Poisson bracket on $C^\infty(B^n)$ using the above formula with $g \in G$ replaced by $b \in B$.

Let $\varphi \in C^\infty(G^n)$. Let π be the Poisson tensor on G^n considered as an element of $Hom(T^*(G^n), T(G^n))$. We have

Definition 4.6 *The Hamiltonian vector field associated to φ is the vector field $X_\phi \in C^\infty(G^n, T(G^n))$ given by*

$$X_\varphi = \pi d\varphi$$

We will need a formula for X_ϕ.

Lemma 4.7 *Let $g = (g_1, ..., g_n)$. Then $X_\varphi(g) = (X_1(g), ..., X_n(g))$ where*

$$X_i(g) = \frac{1}{2}[dL_{g_i}\mathcal{R}D_i\varphi(g) - dR_{g_i}\mathcal{R}D_i'\varphi(g)] \quad .$$

Proof: We will use the formula

$$\{\varphi, \psi\} = -d\psi(X_\varphi) = -\langle X_\varphi, \nabla\psi\rangle$$

Here ∇ is the gradient with respect to $\langle , \rangle$ on $\mathfrak{g}^n$, see above, hence $\nabla\psi = (\nabla_1\psi, ..., \nabla_n\psi)$. We have

$$\{\varphi, \psi\}(g) = \frac{1}{2}\sum_{i=1}^{n}[\langle \mathcal{R}D_i'\varphi(g), D_i'\psi(g)\rangle - \langle \mathcal{R}D_i\varphi(g), D_i\psi(g)\rangle] =$$

$$\frac{1}{2}\sum_{i=1}^{n}[\langle \mathcal{R}D_i'\varphi(g), dL_{g_i}^{-1}\nabla_i\psi(g)\rangle - \langle \mathcal{R}D_i\varphi(g), dR_{g_i}^{-1}\nabla_i\psi(g)\rangle] =$$

$$\frac{1}{2}\sum_{i=1}^{n}[\langle dL_{g_i}\mathcal{R}D_i'\varphi(g), \nabla_i\psi(g)\rangle - \langle dR_{g_i}\mathcal{R}D_i\varphi(g), \nabla_i\psi(g)\rangle] =$$

$$\frac{1}{2}\langle (X_1(g), ..., X_n(g)), (\nabla_1\psi(g), ..., \nabla_n\psi(g))\rangle = \frac{1}{2}\langle X_\varphi(g), \nabla\psi(g)\rangle \qquad \square$$

Remark 4.8 *Since $w(b)$ is tangent to B^n the field $X_\varphi(b)$ will also be tangent to B^n.*

5 The dressing action of K on B^n and the action on n-gons in G/K

The basic reference for this section is [3]. In that paper the authors take $n = 2$ and write $G = KB$. We will leave to the reader the task of comparing our formulae with theirs.

In what follows we let ρ_B, ρ_K be the projections relative to the decomposition $G = BK$. For the next theorem (in the case $n = 2$) see [3, Formula (2.15)].

Theorem 5.1 *There is a Poisson action of K on the Poisson manifold B^n given by*

$$k \cdot (b_1, ..., b_n) = (b_1', ..., b_n')$$

with $b_i' = \rho_B(\rho_K(kb_1 \dots b_{i-1})b_i), 1 \leq i \leq n$.

Definition 5.2 *The above left action is called the* **(right) dressing action** *of K on B^n.*

We will also need the formula for the infinitesimal dressing action of $\mathfrak{k}$ on B^n. This action is given for $x \in \mathfrak{k}$ by

$$x \cdot (b_1, \dots , b_n) = (\xi_1, ..., \xi_n) \in T_b(B^n)$$

with $\xi_i = dL_{b_i} \rho_{\mathfrak{b}} Ad_{b_i^{-1}} \rho_{\mathfrak{k}} Ad_{(b_1 \cdots b_{i-1})^{-1}} x$. Note that $\xi_i \in T_{b_i}(B)$.

Remark 5.3 *In order to pass from the K-action to the $\mathfrak{k}$-action observe that $\rho_K(bg) = \rho_K(g)$ and $\rho_B(bg) = b\rho_B(g)$. Accordingly we may rewrite the K-action as $k \cdot (b_1, ..., b_n) = (b_1', ..., b_n')$ with*

$$b_i' = b_i \rho_B(b_i^{-1} \rho_K((b_1 \cdots b_{i-1})^{-1} kb_1 \cdots b_{i-1})b_i), 1 \leq i \leq n$$

Since B acts simply-transitively on G/K we have

Lemma 5.4 *(i) The map $\Phi : B^n \to Pol_n(X, *)$ given by*

$$\Phi(b_1, ..., b_n) = (*, b_1 *, ..., b_1 \cdots b_n *)$$

is a diffeomorphism.

(ii) The map Φ induces a diffeomorphism from $\{b \in B^n : b_1 \cdots b_n = 1\}$ onto $CPol_n(X, *)$.

We now have

Lemma 5.5 *Φ is a K-equivariant diffeomorphism where K acts on B^n by the dressing action and on $Pol_n(X, *)$ by the natural (diagonal) action.*

Proof: Let $k \cdot (b_1, ..., b_n) = (b_1'', ..., b_n'')$ be the pull-back to B^n of the action of K on $Pol_n(X, *)$. Then

$$\begin{aligned} b_1'' * &= kb_1 * \\ b_1'' b_2'' * &= kb_1 b_2 * \\ &\vdots \\ b_1'' \cdots b_n'' * &= kb_1 \cdots b_n * \end{aligned}$$

We obtain

$$b_1'' \cdots b_i'' = \rho_B(kb_1 \cdots b_i)$$

$$\begin{aligned} b_i'' = \rho_B((b_1'' \cdots b_{i-1}'')^{-1} k(b_1 \cdots b_i)) &= \rho_B((\rho_B(b_1'' \cdots b_{i-1}'))^{-1} kb_1 \cdots b_{i-1} b_i) = \\ \rho_B((\rho_B(kb_1 \cdots b_{i-1}))^{-1} kb_1 \cdots b_{i-1} b_i) &= \rho_B(\rho_K(kb_1 \cdots b_{i-1})b_i) \qquad \square \end{aligned}$$

There is another formula for the dressing action of K on B^n that will be useful.

Lemma 5.6 *With the above notation*

$$b'_i = \rho_B(\underbrace{\rho_K(\dots(\rho_K}_{i-1\ times}(kb_1)b_2)\cdots b_{i-1})b_i)$$

Proof: Induction on i. $\square$

We obtain a corresponding formula for the infinitesimal dressing action of $\mathfrak{k}$ on B^n.

Lemma 5.7 $x \cdot (b_1, ..., b_n) = (\xi_1, ..., \xi_n)$ *where*

$$\xi_i = dL_{b_i}\rho_{\mathfrak{b}} Ad_{b_i^{-1}}\rho_{\mathfrak{k}} Ad_{b_{i-1}^{-1}} \cdots \rho_{\mathfrak{k}} Ad_{b_1^{-1}} x \quad .$$

We now draw an important consequence.

Lemma 5.8 *The map Φ induces a diffeomorphism between the dressing orbit Kb of $b \in B^n$ and the configuration space of open based n-gon linkages $\tilde{N}_w(X, *)$, where $w = (w_1, .., w_n)$ and w_i is the image in the Weyl chamber W of the geodesic segment $[*, b_i*]$.*

Proof: Let $b \in B^n$ be given. Then $\Phi(b) = (*, b_1*, ..., b_1 \dots b_n*)$. The K-orbit of $\Phi(b)$ is

$$[*, kb_1*, ..., k(b_1 \cdots b_n)*]$$

The i-th edge e_i of $\Phi(b)$ is the geodesic segment joining $kb_1 \dots b_{i-1}*$ to $kb_1 \cdots b_i*$. Clearly this is congruent (by $kb_1 \cdots b_{i-1}$) to the segment connecting $*$ to b_i*. $\square$

Corollary 5.9 *The symplectic leaves of B^n map to the configuration spaces $\tilde{N}_w(X, *)$ under Φ.*

Proof: The symplectic leaves of the dressing action are the orbits of the dressing action. $\square$

We let B^n_w denote $\Phi^{-1}(\tilde{N}_w(X, *))$.

6 The moduli space M_w as a symplectic quotient

We have seen that Φ induces a symplectic diffeomorphism from $\{(b_1, ..., b_n) \in B_t : b_1 \dots b_n = 1\}/K$ to the moduli space M_w of closed n-gon linkages in X modulo isometry. In this section we will prove that the map $\varphi : B^n \to B$ given by $\varphi(b_1, \dots, b_n) = b_1 \cdots b_n$ is a momentum map for the (dressing) K action on B^n. Hence M_w is a symplectic quotient, in particular it is a symplectic manifold if 1 is a regular value by Lemma 6.4 below.

The definition of a momentum map for a Poisson action of a Poisson Lie group was given in [5].

Definition 6.1 *Suppose that K is a Poisson Lie group, (M, π) is a Poisson manifold and we are given a Poisson action $K \times M \to M$. Let $x \in \mathfrak{k}$, α_x be the extension of $x \in \mathfrak{k} = (\mathfrak{k}^*)^*$ to a right-invariant 1-form on K^* and $\hat{x}$ be the induced vector field on M. Then a map $\varphi : M \to K^*$ is a* **momentum map** *if it satisfies the equation*

$$-\pi\varphi^*\alpha_x = \hat{x}$$

Remark 6.2 *For the definition of K^*, the* **dual Poisson Lie group***, see [2, Chapter 1]. In our case $K^* = B$.*

Lemma 6.3 *Suppose that M is a symplectic manifold, K is a Poisson Lie group and $K \times M \to M$ is a Poisson action with an equivariant momentum map $\varphi : M \to K^*$. Assume 1 is a regular value of φ. Then $\varphi^{-1}(1)/K$ is a symplectic manifold with the symplectic structure given by taking quotient and restriction of the symplectic structure on M.*

Proof: Let ω be the symplectic form and $m \in \varphi(1) \subset M$. Let V_m be a complement to $T_m\varphi^{-1}(1)$ in T_mM. Let $\mathfrak{k} \cdot m \subset T_mM$ be the tangent space to the orbit $K \cdot m$. Hence $\mathfrak{k}_m = \{\hat{x}(m) : x \in \mathfrak{k}\}$. We first prove the identity

$$\omega_m \hat{x}(m) = -\varphi^* \alpha_x|_m$$

Here we use ω_m to denote the induced map $T_mM \to T_m^*M$ as well as the symplectic form evaluated at m. Indeed we have the identity

$$-\pi \varphi^* \alpha_x = \hat{x}$$

Applying ω we get

$$\omega \hat{x} = -\varphi^* \alpha_x$$

We claim that if $m \in \varphi^{-1}(1)$ the $\mathfrak{k} \cdot m$ is orthogonal (under ω_m) to $T_m(\varphi^{-1}(1))$, in particular it is totally-isotropic. Let $\hat{x}(m) \in \mathfrak{k} \cdot m$ and $u \in T_m(\varphi^{-1}(1))$. Then

$$\omega_m(\hat{x}(m), u) = -\varphi^* \alpha_x|_m(u) = -\alpha_x|_m(d\varphi_m u)$$

But $d\varphi_m u = 0$ and the claim is proved. Hence the restriction of ω_m to $T_m(\varphi^{-1}(1))$ descends to $T_m(\varphi^{-1}(1))/\mathfrak{k} \cdot m = T_{K \cdot m}(\varphi^{-1}(1)/K)$. We now prove that the induced form is nondegenerate.

To this end we claim that $\mathfrak{k} \cdot m$ and V_m are dually paired by ω_m. We draw two conclusions from the hypothesis that 1 is a regular value for φ. First by [3, Lemma 4.2] the map $\mathfrak{k} \to \mathfrak{k} \cdot m$ given by $x \mapsto \hat{x}(m)$ is an isomorphism. Second, $d\varphi_m : V_m \to \mathfrak{k}^*$ is an isomorphism. Let $\{x_1, ..., x_N\}$ be a basis for $\mathfrak{k}$, whence $\{\hat{x}_1(m), ..., \hat{x}_N(m)\}$ is a basis for $\mathfrak{k} \cdot m$. We want to find a basis $\{v_1, ..., v_N\}$ for V_m so that $\omega_m(\hat{x}_i(m), v_j) = \delta_{ij}$. Choose a basis $\{v_1, ..., v_N\}$ for V_m such that $\{d\varphi_m v_1, ..., d\varphi_m v_N\} \subset \mathfrak{k}^*$ is dual to $\{x_1, ..., x_N\}$. Then

$$\omega_m(\hat{x}_i(m), v_j) = -\varphi^* \alpha_{x_i}|_m(v_j) = -\alpha_{x_i}|_m(d\varphi v_j) = -d\varphi v_j(x_i) = -\delta_{ij}$$

As a consequence of the previous claim the restriction of ω_m to $\mathfrak{k} \cdot m \oplus V_m$ is nondegenerate. Hence the orthogonal complement $(\mathfrak{k} \cdot m \oplus V_m)^\perp$ is a complement to $\mathfrak{k} \cdot m \oplus V_m$ and $\omega_m|(\mathfrak{k} \cdot m \oplus V_m)^\perp$ is nondegenerate. But then $(\mathfrak{k} \cdot m \oplus V_m)^\perp$ maps isomorphically to $T_{K \cdot m}(\varphi^{-1}(1)/K)$. $\square$

We now begin the proof that φ is a momentum map for the dressing action of K on B^n. We will need some notation. Let $x \in \mathfrak{k} = \mathfrak{b}^*$. Recall that α_x is the extension of x to a right-invariant 1-form on B. Thus if $\zeta \in T_b(B)$ we have

$$\alpha_x|_b(\zeta) = \langle dR_b x, \zeta \rangle$$

Lemma 6.4 $-\pi \alpha_x|_b = dL_b \rho_{\mathfrak{b}} Ad_{b^{-1}} x$.

Proof: Let $y \in \mathfrak{k}$. It suffices to prove that

$$\pi|_b(\alpha_y(b), \alpha_x(b)) = -\alpha_y|_b(dL_b \rho_{\mathfrak{b}} Ad_{b^{-1}} x)$$

Now

$$\pi|_b(\alpha_y(b), \alpha_x(b)) = \pi|_b(dR_b y, dR_b x) = \pi|_e(y, x) = \langle \rho_{\mathfrak{k}}(Ad_{b^{-1}} y), \rho_{\mathfrak{b}}(Ad_{b^{-1}} x)\rangle$$

according to Lemma 4.5. Also

$$-\alpha_y|_b(dL_b \rho_{\mathfrak{b}} Ad_{b^{-1}} x) = -\langle dR_b y, dL_b \rho_{\mathfrak{b}} Ad_{b^{-1}} x\rangle =$$

$$= -\langle Ad_{b^{-1}} y, \rho_{\mathfrak{b}} Ad_{b^{-1}} x\rangle = -\langle \rho_{\mathfrak{k}} Ad_{b^{-1}} y, \rho_{\mathfrak{b}} Ad_{b^{-1}} x\rangle$$

□

Lemma 6.5 *$\varphi^*\alpha_x|_b = (\alpha_{x_1}|_{b_1}, \alpha_{x_2}|_{b_2}, \dots, \alpha_{x_n}|_{b_n})$, where $x_1 = x$ and $x_i = \rho_{\mathfrak{k}}(Ad_{(b_1 \cdots b_{i-1})^{-1}} x)$, $2 \le i \le n$.*

Proof: We will use the following formula (the product rule). Let $b = (b_1, ..., b_n)$ and $\zeta = (\zeta_1, ..., \zeta_n) \in T_b(B^n)$. Then

$$d\varphi_b(\zeta) = (dR_{b_2 \dots b_n}\zeta_1, dL_{b_1} dR_{b_3 \dots b_n}\zeta_2, \dots, dL_{b_1 \dots b_{n-1}}\zeta_n)$$

Hence

$$(\varphi^*\alpha_x)|_b(\zeta) = \alpha_x|_{b_1 \dots b_n}(dR_{b_2 \dots b_n}\zeta_1,\ dL_{b_1} dR_{b_3 \dots b_n}\zeta_2,\ \dots,\ dL_{b_1 \dots b_{n-1}}\zeta_n) =$$

$$\langle dR_{b_1 \dots b_n} x, dR_{b_2 \dots b_n}\zeta_1\rangle + \langle dR_{b_1 \dots b_n} x, dL_{b_1} dR_{b_3 \dots b_n}\zeta_2\rangle + \dots \langle dR_{b_1 \dots b_n}, dL_{b_1 \dots b_{n-1}}\zeta_n\rangle$$

$$= \langle dR_{b_1} x, \zeta_1\rangle + \langle dR_{b_2} Ad_{b_1^{-1}} x, \zeta_2\rangle + \dots + \langle dR_{b_n} Ad_{(b_1 \dots b_{n-1})^{-1}} x, \zeta_n\rangle =$$

$$\langle x, dR_{b_1^{-1}}\zeta_1\rangle + \dots + \langle Ad_{(b_1 \dots b_{n-1})^{-1}} x, dR_{b_n}^{-1}\zeta_n\rangle =$$

$$\langle \rho_{\mathfrak{k}}(x), dR_{b_1^{-1}}\zeta_1\rangle + \dots + \langle \rho_{\mathfrak{k}}(Ad_{(b_1 \dots b_{n-1})^{-1}} x), dR_{b_n}^{-1}\zeta_n\rangle.$$

□

Proposition 6.6 *φ is an equivariant momentum map for the right dressing action of K on B^n.*

Proof: To show that φ is a momentum map we have to check that

$$-\pi\mu^*\alpha_x|_b = x \cdot (b_1, ..., b_n)$$

But $-\pi\mu^*\alpha_x = (-\pi\alpha_{x_1}, \dots, -\pi\alpha_{x_n})$ and the result follows from the previous two lemmas. To show that φ is equivariant we have to check that $\varphi(k \cdot (b_1, ..., b_n)) = k \cdot b_1 ... b_n$. This is obvious from the point of view of polygons. □

As a consequence of the above proposition we obtain

Theorem 6.7 *The map Φ carries the symplectic quotient $(\varphi|_{B^n_w})^{-1}(1)/K$ symplectomorphically to the moduli space of n-gon linkages $M_w(X)$.*

7 The bending Hamiltonians

In this section we will compute the Hamiltonian vector fields X_{f_j} of the functions

$$f_j(b) = tr((b_1 \cdots b_j)(b_1 \cdots b_j)^*), 1 \leq j \leq n$$

We will assume $G = SL_n(\mathbb{C})$. The material of this section is due to T. Treeloar.

We will use the following notation. If $A \in M_n(\mathbb{C})$ then $A^0 = A - \frac{1}{n}tr(A)I$ will be its projection to the traceless matrices.

Theorem 7.1 *(T. Treeloar) Define $F_j : B^n \to \mathfrak{k}$ by*

$$F_j(b) = \sqrt{-1}[(b_1 \cdots b_j)(b_1 \cdots b_j)^*]^0$$

Then $X_{f_j} = 2F_j(b) \cdot (b_1, ..., b_n)$ where $\cdot$ is the infinitesimal dressing action of §5.

Proof: It will be convenient to work on G^n and then restrict to B^n. By the formula for X_φ of Lemma 4.7 it suffices to compute $D_i f_j$ and $D_i' f_j$. We recall that

$$D_i'\varphi(g) = Ad_{g_i^{-1}} D_i \varphi(g)$$

hence it suffices to compute $D_i f_j(g)$. We first reduce to computing $D_1 f_j$ by

Lemma 7.2 $D_i f_j(g) = Ad_{(g_1 \cdots g_{i-1})^{-1}} D_1 f_j(g)$.

Proof: By definition

$$\langle D_i f_j(g), \nu \rangle = \frac{d}{dt}|_{t=0} f_j(g_1, ..., e^{t\nu} g_i, ..., g_n)$$

But it is elementary that

$$f_j(g_1, ..., e^{t\nu} g_i, ..., g_n) = f_j((Ad_{(g_1 \cdots g_{i-1})^{-1}} e^{t\nu} g_1, ..., g_n)$$

Differentiating at $t = 0$ we obtain

$$\langle D_i f_j(g), \nu \rangle = \langle D_1 f_j(g), Ad_{(g_1 \cdots g_{i-1})^{-1}} \nu \rangle = \langle Ad_{(g_1 \cdots g_{i-1})^{-1}} D_1 f_j(g), \nu \rangle \qquad \square$$

We next have

Lemma 7.3 $D_1 f_j(g) = 2F_j(g)$.

Proof: By definition

$$\langle D_1 f_j(g), \nu \rangle = \frac{d}{dt}|_{t=0} tr[(e^{t\nu} g_1 \cdots g_j)(e^{t\nu} g_1 \cdots g_j)^*] =$$

$$tr[(\nu g_1 \cdots g_j)(g_1 \cdots g_j)^* + (g_1 \cdots g_j)(\nu g_1 \cdots g_j)^*] =$$

$$tr[(\nu g_1 \cdots g_j)(g_1 \cdots g_j)^*] + tr[(\nu g_1 \cdots g_j)(g_1 \cdots g_j)^*]^* =$$

$$tr[(\nu g_1 \cdots g_j)(g_1 \cdots g_j)^*] + \overline{tr[(\nu g_1 \cdots g_j)(g_1 \cdots g_j)^*]} =$$

$$2Re tr[(\nu g_1 \cdots g_j)(g_1 \cdots g_j)^*] =$$

$$2Im\sqrt{-1} tr[\nu (g_1 \cdots g_j)(g_1 \cdots g_j)^*]$$

Since $\nu \in sl_n(\mathbb{C})$ we may replace

$$(g_1 \cdots g_j)(g_1 \cdots g_j)^*$$

by its traceless projection

$$[(g_1 \cdots g_j)(g_1 \cdots g_j)^*]^0 \quad .$$

Since tr is complex bilinear we obtain

$$\langle D_1 f_j(g), \nu \rangle = 2Imtr[\nu\sqrt{-1}[(g_1 \cdots g_j)(g_1 \cdots g_j)^*]^0] = \langle 2\sqrt{-1}[(g_1 \cdots g_j)(g_1 \cdots g_j)^*]^0, \nu \rangle$$

□

Now we restrict to B^n and substitute into our formula for $X_{f_j}(b)$ in Lemma 4.7. We obtain

Lemma 7.4 $-\frac{1}{2}(X_{f_j})_i = (D_1 f_j(b) \cdot (b_1, ..., b_n))_i =$ *the i-th component of the infinitesimal dressing action of* $D_1 f_j(b) \in \mathfrak{k}$.

Proof: By Lemma 4.7 we have

$$(X_{f_j})_i = \frac{1}{2}[dL_{b_i} R D_i f_j(b) - dR_{b_i} \mathcal{R} D_i' f_j(b)] =$$

$$\frac{1}{2}[dR_{b_i} \mathcal{R} Ad_{(b_1 \cdots b_{i-1})^{-1}} D_1 f_j(b) - dL_{b_i} \mathcal{R} Ad_{b_i} Ad_{(b_1 \cdots b_{i-1})^{-1}} D_1 f_j(b)]$$

We write

$$Ad_{(b_1 \cdots b_{i-1})^{-1}} D_1 f_j(b) = X_1 + \eta_1$$

with $X_1 \in \mathfrak{k}$ and $\eta \in \mathfrak{b}$. Hence

$$X_1 = \rho_{\mathfrak{k}}(Ad_{(b_1 \cdots b_{i-1})^{-1}} D_1 f_j(b))$$

$$\eta_1 = \rho_{\mathfrak{b}}(Ad_{(b_1 \cdots b_{i-1})^{-1}} D_1 f_j(b))$$

Then

$$\mathcal{R} Ad_{(b_1 \cdots b_{i-1})^{-1}} D_1 f_j(b) = X_1 - \eta_1$$

We write

$$Ad_{b_i^{-1}} X_1 = X_2 + \eta_2, X_2 \in \mathfrak{k}, \eta_2 \in \mathfrak{b}$$

Then $X_2 = \rho_{\mathfrak{k}}(Ad_{b_i^{-1}} X_1), \eta_2 = \rho_{\mathfrak{b}}(Ad_{b_i^{-1}} X_1)$ and

$$Ad_{b_i^{-1}}(X_1 + \eta_1) = X_2 + \eta_2 + Ad_{b_i^{-1}} \eta_1$$

Hence

$$\mathcal{R} Ad_{b_i^{-1}} Ad_{(b_1 \cdots b_{i-1})^{-1}} D_1 f_j(b) = X_2 - \eta_2 - Ad_{b_i^{-1}} \eta_1 =$$

$$X_2 + \eta_2 - 2\eta_2 - Ad_{b_i^{-1}} \eta_1 =$$

$$Ad_{b_i^{-1}} X_1 - 2\eta_2 - Ad_{b_i^{-1}} \eta_1$$

Hence

$$(X_{f_j})_i = \frac{1}{2}[dR_{b_i} X_1 - dR_{b_i} \eta_1 - dL_{b_i} Ad_{b_i^{-1}} X_1 + 2dL_{b_i} \eta_2 + dL_{b_i} Ad_{b_i^{-1}} \eta_1]$$

But $dL_{b_i} Ad_{b_i^{-1}} = dR_{b_i}$ and we obtain

$$(X_{f_j})_i = dL_{b_i} \eta_2$$

Since $\eta_2 = \rho_{\mathfrak{b}}(Ad_{b_i^{-1}} \rho_{\mathfrak{k}}(Ad_{(b_1 \cdots b_{i-1})^{-1}} D_1 f_j(b)))$ the lemma follows. □

With this Theorem 7.1 is proved. □

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