TOPOLOGICAL ASPECTS OF KLEINIAN GROUPS IN SEVERAL DIMENSIONS

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0. Introduction and basic definitions.

Let $\mathbf{Mob}(\mathbb{S}^n)$ be the group of all orientation-preserving Mobius transformations of the n-sphere $\mathbb{S}^n = \overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$. This group is isomorphic to the connected component of 1 in the Lorentz group $SO(n+1,1)$. The fixed- point set of $\gamma \in \mathbf{Mob}(\mathbb{S}^n)$ is denoted by $Fix(\gamma) = \{x \in \mathbb{S}^n : \gamma(x) = x\}$. For a group $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ the *discontinuity set* $\Omega(\Gamma)$ is equal to

$\{x \in \mathbb{S}^n :$ the point x possesses a neighborhood $U(x)$ such that the intersection $U(x) \cap \gamma(U(x))$ is empty for all but finite elements $\gamma \in \Gamma\}$.

We shall denote the factor $\Omega(G)/G$ by $M(G)$. The complement $\mathbb{S}^n \backslash \Omega(\Gamma) = \Lambda(\Gamma)$ is the *limit set* of the group Γ. A subgroup $\Gamma \subset \mathbf{Mob}(\mathbb{S}^n)$ is called *Kleinian* if $\Omega(\Gamma)$ isn't empty. So, Kleinian groups are discontinuous groups of Mobius transformations; we do not require $\Lambda(\Gamma)$ to be infinite. In this paper we shall always assume Kleinian groups to be finitely generated.

Any Mobius transformation $\gamma \in \mathbf{Mob}(\mathbb{S}^n)$ may be extended to an element $\tilde{\gamma} \in \mathbf{Mob}(\mathbb{S}^{n+1})$. This extension agrees with the embedding $SO(n+1,1) \subset SO(n+2,1)$. The element $\tilde{\gamma}$ has a closed invariant ball $\mathbb{B}^{n+1} \subset \mathbb{S}^{n+1}$ with the boundary $\mathbb{S}^n$; the interior of $\mathbb{B}^{n+1}$ plays the role of a model for the hyperbolic space $\mathbb{H}^{n+1}$.

An element γ is said to be *loxodromic* if $Fix(\tilde{\gamma}) \cap \mathbb{B}^{n+1} = \{p \neq q\} \subset \mathbb{S}^n$. An element γ is said to be *parabolic* if $Fix(\tilde{\gamma}) \cap \mathbb{B}^{n+1} = \{p\} \subset \mathbb{S}^n$; and γ is said to be elliptic in the either case: $Fix(\tilde{\gamma}) \cap\ int\mathbb{B}^{n+1} \neq \emptyset$.

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If G is a Kleinian group then G-conjugacy class of a maximal parabolic subgroup of G is called to be a cusp. Geometrically cusps 1-1 correspond to connected components of the thin part of $\mathbb{H}^{n+1}/G$ where the injectivity radius isn't bounded from zero.

A fundamental set for the Kleinian group G is a subset Φ of $\Omega(G)$ such that the orbit $G \cdot \Phi$ coincides with $\Omega(G)$ and

$g(\Phi) \cap \Phi = \emptyset$ for any $g \in G \backslash \{1\}$.

The limit set $\Lambda(G)$ consists either of 0, 1, 2 or infinitely many points; if $\Lambda(G)$ is infinite then it's a compact perfect set. A Kleinian group is elementary iff its limit set is finite.

Let N_G be the convex hull in $\mathbb{H}^{n+1}$ of the limit set of a discrete group $G \subset$**Mob**$(\mathbb{S}^n)$. The set N_G is called to be the *Nielsen's convex hull* of the group G and it plays very important role in the theory of Kleinian groups. This set is closed, convex and invariant under the action of the group G. However, its dimension can be less than $n + 1$ (for the simplest example take an elementary Kleinian group).

For some $\epsilon > 0$ consider an ϵ-neighborhood $N_G(\epsilon)$ of N_G in $\mathbb{H}^{n+1}$. Then we can form the factor-space $N_G(\epsilon)/G$. The group G is called to be *geometrically finite* if $\mathrm{vol}(N_G(\epsilon))/G < \infty$. There are several other definitions of geometrical finiteness (see e.g. [Bo], [Bon], [Tu 1]).

Criterion for geometrical finiteness (A.Beardon & B.Maskit [Be M], B.Bowditch [Bo]). The group G is geometrically finite iff any point of $\Lambda(G)$ is either a *point of approximation* or a *cusp-point*.

The point $x \in \Lambda(G)$ is a point of approximation if for every $y \in \Lambda(G) - \{x\}$ there exists a sequence $g_k \in G$ such that $\lim_{k\to\infty} g_k(z) = y$ for every $z \in \Lambda(G) - \{x\}$.

The point $x \in \Lambda(G)$ is a cusp-point if x is a fixed point of some parabolic subgroup $P \subset G$ such that $(\Lambda(G) - \{x\})/P$ is compact.

For example, if G has no parabolic elements then geometrical finiteness is equivalent to the assumption that $(\mathbb{H}^{n+1} \cup \Omega(G))/G \cong N_G(\epsilon)/G$ is compact. In this case the group G is called *convex-cocompact*. If $n = 2$ then the classical definition of geometrical finiteness requires existence of a convex finite-sided fundamental polyhedron for the action of G in $\mathbb{H}^3$, however in the higher dimensions such condition is questionable. In this paper we shall also discuss an object closely related to the theory of Kleinian groups, namely *flat conformal structures*. Classically, flat conformal structure K on a manifold M is just a conformal class of conformally-Euclidean Riemannian metrics (i.e. locally $ds^2 = \rho(x)|dx|^2$, however, globally M need not to have any Euclidean metric). From the topological point of view, K is a conformal class of smooth structures on M with conformal transition maps. Thus, the standard flat conformal structure on $\mathbb{S}^n$ projects to a flat conformal structure K_G on the manifold $M(G) = \Omega(G)/G$ if G acts freely on its discontinuity domain. The structures of such kind are called *uniformizable* as a reminiscent of the classical uniformization theorem for Riemannian surfaces. See for instance [KAG], [Kul 3] for details.

Sources of Kleinian groups:

(a) Arithmetic groups and their subgroups.

(b) Poincare theorem about fundamental polyhedra.

(c) Klein– Maskit Combination Theorems.

(d) Small deformations of Kleinian groups.
(e) "Large" deformations (limits of sequences of Kleinian groups).

The source (b), in principle, is the most general. The Poincare theorem about fundamental polyhedra (see [Mk 1], [A 1]) tells us that given a polyhedron Φ in $\mathbb{H}^{n+1}$ and a family of isometries $g_1, g_2, ..., g_k, ...$ pairing the faces of Φ, under certain condition on this "initial data" the group G generated by $g_1, g_2, ..., g_k, ...$ is discrete and Φ is a fundamental polyhedron for the group G. On another hand, each Kleinian group has a fundamental polyhedron (for example, Dirichlet or Ford fundamental domain). However, in practice, it's not so easy to use this Theorem, especially if Φ has infinitely many sides. Another problem is that almost always the choice of Φ is rather the question of skill than of science.

The Klein– Maskit Combination Theorems (see [Mk 1], [KAG]) provide conditions that guarantee that a group G generated by two Kleinian subgroups G_1, G_2 is again Kleinian and has some special algebraic structure (amalgamated free product or HNN-extension) as well as the factor-manifold of the group G is obtained by some "cut-and-paste" operation with the manifolds $\Omega(G_1)/G_1$, $\Omega(G_2)/G_2$.

The most regular way to construct Kleinian groups is based on arithmetic examples (for definition see [Rag]).

A good example of the application of combination of (a) and (c) is the example of M.Gromov & I.Piatetski-Shapiro [GP] of non-arithmetic lattices in $SO(n, 1)$. Starting with two arithmetic groups Γ_j $(j = 1, 2)$ they "cut these groups in half", take $G_j \subset \Gamma_j$, and then– combine G_j via Maskit Combination. The construction in [GT] is an application of (a), (d), (c).

The most sophisticated (to the current date) construction of Kleinian groups is provided by Thurston's hyperbolization theorem, but still it's, basically, a combination of (d), (e) and (c).

Thurston's Theorem about hyperbolic Dehn surgery is a beautiful example of (e).

There is a potential sixth source of Kleinian groups in higher dimensions: monodromy of differential equations, but, to the my best knowledge, all that was done here is concerned with discrete subgroups of $PGL(n, \mathbb{C})$.

Equivalence classes of Kleinian groups. Isomorphic Kleinian groups can be considered up to different equivalence relations

(i) **Topological**: for the pair of Kleinian groups $G_1, G_2 \subset$**Mob**$(\mathbb{S}^n)$ there exists a homeomorphism $f : \mathbb{S}^n \to \mathbb{S}^n$ such that $fG_1f^{-1} = G_2$.

(ii) **Weak topological equivalence**: the homeomorphism f is defined just on the limit sets.

(iii) **Quasiconformal equivalence**: in (i) one can find a quasiconformal homeomorphism.

(iv) **Strong topological equivalence**: in (i) there exists actually a continuous family of homeomorphisms h_t such that:

$h_0 = id$; $h_tG_1h_t^{-1} \subset$**Mob**$(\mathbb{S}^n)$; $h_1G_1h_1^{-1} = G_2$.

(v) **Strong quasiconformal equivalence**: in (iv) all homeomorphisms are quasiconformal.

Suppose that the both groups G_j are geometrically finite and the isomorphism $\varphi : G_1 \to G_2$ preserves the *type* of elements (i.e. the images of loxodromic elements are loxodromic etc.). Then

THEOREM 1 (P.Tukia [Tu 1]). Under the conditions above suppose also that there exists a (quasiconformal) homeomorphism $f_0 : D_1 \to D_2$ conjugating actions of G_1 and G_2 on the invariant subsets D_j of their discontinuity domains (D_j can be empty). Then the isomorphism φ can be realized by an equivalence (ii), i.e. $fgf^{-1} = \varphi(g)$ for all $g \in G_1$, where f extends f_0 to a (quasiconformal) homeomorphism of $D_1 \cup \Lambda(G_1)$ (cf. [Mar 1]).

Quasiconformal vs. topological. Suppose that two Kleinian groups G_1, G_2 are topologically conjugate in $\mathbb{S}^n$. Does it imply that they can be conjugate via quasiconformal homeomorphism ?
If $n = 1$ then the answer is "yes", the answer is positive also if $n \neq 4$ and $M(G_1)$ is compact (this is a particular case of the result of D.Sullivan [Su 2]). If $n = 4$ and $M(G_1)$ is compact then the situation is unclear but one can expect the negative answer (see [DS]). If $n = 2$ then the answer is again positive for geometrically finite groups (A.Marden [Mar 1]). Moreover, it follows that there exists even strong quasiconformal equivalence between G_1, G_2. But for geometrically infinite groups the question is open and it is closely related to the famous Bers' Conjecture:

every totally degenerate group is a limit of quasifuchsian ones.

The weak topological equivalence implies that the *ending laminations* of the groups under consideration are the same. If the radii of injectivity for the both manifolds $\mathbb{H}^3/G_j$ are bounded away from zero and G_j aren't decomposable in non-trivial free products, then Y.Minsky succeeded to prove that $\mathbb{H}^3/G_1$, $\mathbb{H}^3/G_2$ are quasiisometric. So, in this case the groups G_j are quasiconformally equivalent.

However if we consider geometrically finite Kleinian groups in $\mathbf{Mob}(\mathbb{S}^3)$ with parabolic elements then the answer turns to be negative, for example: two parabolic transformations are

always conjugate topologically , but not always quasiconformally .

The question: "does (i) implies (iv) or (iii) implies (v) ?"

seems to be open in higher dimensions in spite of the fact that the space of flat conformal structures on the given manifold can be disconnected (see [Ka 5] and Section 2), but one can expect the failure of these implications even for convex-cocompact groups.

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1. Groups with totally disconnected limit sets.

Suppose that G isn't elementary and $\Lambda(G)$ is totally disconnected (i.e. $\Lambda(G)$ is a discontinuum), then the classical theorem of general topology tells us that $\Lambda(G)$ is homeomorphic to a Cantor set $K \subset [0, 1]$. The question is:

How K can be embedded in $\mathbb{S}^n$?

If $n \leq 2$ then for every totally disconnected perfect compact $K' \subset \mathbb{S}^n$ there exists a homeomorphism $f : \mathbb{S}^n \to \mathbb{S}^n$ which maps K' to the Cantor set $K \subset [0, 1]$. Moreover, if $K' = \Lambda(G)$ and G is free then f can be chosen to be quasiconformal so that fGf^{-1} is a Kleinian group. If $n \geq 3$ then there are examples of "wild" discontinuums K', which means that there is no an ambient homeomorphism which maps K' on a subset of the real line. E.g. the complement to the Antoine discontinuum in $\mathbb{S}^3$ isn't simply-connected. There are slightly more sophisticated examples when $\pi_1(\mathbb{S}^3 - K') = 1$ but K' is still wild. (Hint: one can look at the *local fundamental group* of $\mathbb{S}^3 - K'$ near K').

EXAMPLE 1. Classical Schottky groups.
Suppose that we are given a collection of disjoint closed round balls

$$B_1, B_2, \ldots B_k, B'_1, \ldots B'_k$$

in $\mathbb{S}^n$ and a family of Mobius transformations $g_j : \mathrm{int}(B_j) \to \mathrm{ext}(B'_k)$.

Then the domain $\Phi = \mathbb{S}^n - \bigcup_{j=1}^n (B_j \cup \text{ int } (B'_k))$ is a fundamental domain for the group $G =< g_1, \ldots, g_k >$ which is a free group of rank k. The group G is called to be *a classical Schottky group* and the system of generators $g_1, \ldots, g_k$ is a *classical system of free generators*. A system of free generators for G is called *nonclassical* if it doesn't admit a collection of round balls $\{B_j, B'_j\}$ with the properties above. It's easy to construct a nonclassical system of generators for every classical Schottky group G which preserves a round circle.

EXAMPLE 2. Schottky groups.

Suppose that the closed balls in the definition of the classical Schottky group are just topological balls with locally flat boundary (in the topological sense). Then the group generated by the transformations $g_1, \ldots, g_k$ is called a *Schottky group*. The factor-manifold $M(G) = \Omega(G)/G$ is homeomorphic to the connected sum of k copies of $\mathbb{S}^{n-1} \times \mathbb{S}^1$ and there exists a homeomorphism of $\mathbb{S}^n$ which conjugates G to a Schottky group with invariant circle. Thus, in this case $\Lambda(G)$ is tame (i.e. isn't wild).

Again, the group G generated by $g_1, \ldots, g_k$ is a free group of rank k and (according to V.Chuckrow [Cw]) any system of free generators of a Schottky group admits a collection of topological balls with the properties above.

QUESTION 1. Does there exist a nonclassical Schottky group (i.e. which is nonclassical for any choice of free generators) ?

Any Schottky group which preserves an oriented round circle is classical. The positive (but nonconstructive) answer to Question 1 was done by A.Marden [Mar 2] (if $n \geq 2$), who used arguments appealing to the Teichmuller theory. First (but wrong) example of a nonclassical Schottky group for $n = 2$ was constructed by R.Zarrow [Z]. So, to this date, 100% correct explicit examples of nonclassical Schottky groups exist only for $n \geq 4$ (see below).

EXAMPLE 3. Schottky-type groups.

First take an arbitrary *elementary* Kleinian group H. We can suppose that $H \subset \mathbf{Mob}(\mathbb{S}^n) \cap \mathrm{Aff}(\mathbb{R}^n)$. Then H has a convex fundamental domain D_H in $\mathbb{R}^n$ or $\mathbb{R}^n \setminus \{\text{point}\}$. For example, if H is a cyclic loxodromic group then D_H is bounded by two round spheres.

Suppose that in the definition of Schottky group we allow not only the balls B_j, B'_j as above, but also disjoint images D_j of $\mathbb{S}^n - D_{H_j}$ under homeomorphisms conjugating elementary Kleinian groups. Thus the initial data for Schottky-type groups consists of

(1) union of disjoint balls (whose boundaries are matched by loxodromic transformations);

(2) tangent topological balls (whose boundaries are matched by parabolic transformations which fix the points of tangency);

(3) topological boxes whose faces are matched by parabolic transformations generating almost abelian groups.

Then the group generated by the elementary groups H'_j is called to be a *Schottky-type group* (or, sometimes, an *extended Schottky group*).

According to B.Maskit, Schottky groups acting on $\mathbb{S}^2$ can be characterized by the following nice

THEOREM 2 [Mk 2]. A Kleinian subgroup of $\mathbf{Mob}(\mathbb{S}^2)$ is a Schottky group iff it's free and consists only of loxodromic transformations.

This result was generalized by N.Gusevskii and N.Zindinova as follows:

THEOREM 3 [GZ]. Let G be a Kleinian subgroup of $\mathbf{Mob}(\mathbb{S}^2)$ which is isomorphic to a Schottky- type group via a type-preserving isomorphism. Then G is a Schottky - type group itself.

REMARK 1. In the paper of Rodrigues [Rod] preceding [GZ] Theorem 3 was proved under the assumption that any maximal parabolic subgroup of G is an abelian group of rank 2.

Both theorems are trivial under the assumption that G is geometrically finite, the key point here is that (in the dimension 2) one can prove geometrical finiteness under the mild assumptions above.

If G is a Kleinian subgroup of $\mathbf{Mob}(\mathbb{S}^3)$, then the results above are not longer true, moreover G can be easily geometrically infinite. The example is given by a free purely loxodromic discrete subgroup of $\mathbf{Mob}(\mathbb{S}^2)$, whose limit set is the whole sphere (existence of such groups first was established by V.Chuckrow [Cw]).

THEOREM 4 (R.Kulkarni [Kul 2]). Suppose that a Kleinian group G is torsion-free, finitely generated and $\Lambda(G)$ is a discontinuum. Then G is a free product of *elementary Kleinian groups* H_j.

THEOREM 5 (N.Gusevskii [G 1]). Suppose that G is a torsion-free Kleinian group such that $\Lambda(G)$ is a discontinuum. Then G admits a fundamental domain Φ of the same shape as in Example 3, just B_j, B'_j etc. are **some** manifolds with locally- flat boundary (not necessarily topological balls). In particular, G is geometrically finite.

REMARK 2. An analogous result was established by L.Potyagailo in his thesis under the assumption of geometrical finiteness (see also [P 4]).

The proof of Theorem 5 is based on the following result of M.Brin:

THEOREM 6 [Br]. Let $\tilde{M}$ be an orientable manifold without boundary such that :

(i) $\dim\tilde{M} > 2$;

(ii) every closed embedded 2-sided submanifold of $\tilde{M}$ separates it;

(iii) $\tilde{M}$ admits a properly discontinuous free action of some group G.

Then, for every oriented hypersurface N in $\tilde{M}$ and any open neighborhood U of GN there exists a closed oriented hypersurface $N^* \subset U$ such that: $gN^* \cap N^* = \emptyset$ for every $g \in G - 1$.

This result allows to split inductively the action of the Kleinian group G in free product in the "geometric way". If all N^* are spheres then this decomposition would be of the "Schottky-type".

THEOREM 7 (M.Freedman [F]). Suppose that $n \neq 4$, $G \subset \mathrm{Homeo}(\mathbb{S}^n)$ and $\Lambda(G)$ is a discontinuum with the simply-connected complement. Then G is topologically conjugate to a Schottky group.

THEOREM 8. (M.Freedman, R.Skora [F S 1], [F S 2]). For every $n > 2$ there exists a subgroup $G_n \subset \mathrm{Homeo}(\mathbb{S}^n)$ such that: $\Lambda(G_n)$ is a discontinuum, $\Omega(G_n)/G_n$ is compact, but G_n isn't topologically conjugate to any Kleinian group. In fact G_n isn't conjugate even to any *uniformly quasiconformal group* (see Section 5).

EXAMPLE 4. (M.Bestvina , D.Cooper [B C]). There exists a finitely generated Kleinian group $G \subset \mathrm{Mob}(\mathbb{S}^3)$ which contains parabolic elements and has totally disconnected limit set with non-simply connected complement.

The proof that $\pi_1\Omega(G) \neq 1$ was incomplete. This gap was filled several years later by S.Matsumoto [Mat].

EXAMPLE 5. (S.Matsumoto [Mat], N.Gusevskii [G 2]). There are examples of Kleinian groups G in $\mathrm{Mob}(\mathbb{S}^3)$ without parabolic elements whose limit sets are wild discontinuums.

Extensions of these groups to $\mathrm{Mob}(\mathbb{S}^4)$ are Schottky groups G_4 . Such Schottky groups can't be classical, otherwise the restriction of G_4 to $\mathbb{S}^3$ would be classical Schottky groups too [G 3].

2. Groups whose limit sets are topological spheres.

2.1. We shall discuss here only the case when either $\dim \Lambda(G)$ or $\mathrm{codim}\Lambda(G)$ is equal to 1 (because of the lack of our knowledge in more general case; see, however, Theorem 14).

EXAMPLE 6. *Codimension 1 Fuchsian groups.*
Suppose that $G \subset \mathrm{Mob}(\mathbb{S}^n)$ has an invariant closed round ball $\mathbb{B}$ and $\partial\mathbb{B} = \Lambda(G)$. Then G is called to be a Fuchsian group (of codimension 1 = codimension of its limit set).

(This definition is slightly different from the classical one which allows $\Lambda(G)$ to be a proper subset of $\partial\mathbb{B}$.)

EXAMPLE 7. *Codimension 1 quasi-fuchsian groups.*
Imagine that we have the same situation as in Example 6 but $\mathbb{B}$ isn't a round ball, but still there exists a homeomorphism $f : \mathbb{S}^n \to \mathbb{S}^n$ which conjugates G to a Fuchsian group. Then the group G is a codimension 1 quasi-fuchsian group. If $n > 2$ then it's not so easy to construct nontrivial examples of codimension 1 quasi-fuchsian groups which aren't Fuchsian (see Section 4).

If $n = 2$ then we have the Ahlfors-Bers machinery to produce quasifuchsian groups. If G is isomorphic to $\pi_1(S_g)$ where S_g is a closed surface of genus $g > 1$ then the space of quasifuchsian groups is homeomorphic to $\mathbb{R}^{12(g-1)}$.

THEOREM 9 (B.Maskit [Mk 1], A.Marden , [Mar 1]).

Let $G \subset \mathrm{Mob}(\mathbb{S}^2)$ be such that there are precisely two G-invariant components Ω_1, Ω_2 of $\Omega(G)$. Then G is quasifuchsian and
$M^3(G) = (\mathbb{H}^3 \bigcup \Omega(G))/G$ is homeomorphic to $\Omega_1/G \times [0,1]$.

Our aim is to compare the higher-dimensional case with this Theorem.

2.2. Suppose now that: $\Lambda(G)$ is homeomorphic to $\mathbb{S}^{n-1}$ and has codimension 1 in $\mathbb{S}^n$. Then it's easy to see that $\Omega_1 \bigcup \Omega_2 = \Omega(G)$ and $M(G) = M_1 \bigcup M_2$ (up to transition to a subgroup of index 2 in G), so we shall always suppose below that Ω_j are $G-$ invariant. By the duality we have: $H_*(\Omega_j) \cong H_*(point)$. Therefore, under the condition that $\pi_1(\Omega_j) = 1$, this component is contractible.

THEOREM 10. Suppose that both M_j are compact and $\pi_1(\Omega_j) = 1$. Then: if $n \geq 5$ we have: $M^{n+1}(G)$ is homeomorphic to $M_1 \times [0,1]$. The same is true if $n = 4$ under the assumption: M_1 is homeomorphic to M_2 .

This is just a particular case of results of Farrel and Jones [FJ 1]. Really, let $n \neq 4$, then $M^{n+1}(G)$ is an h-cobordism between two aspherical manifolds M_1 and M_2 whose fundamental groups are isomorphic to fundamental group of a compact convex manifold of (constant) negative sectional curvature. Then this h-cobordism is trivial. If $n = 4$ then one should apply the relative version of [FJ 1] to the pair of manifolds $M^{n+1}(G)$ and $M_1 \times [0,1]$. These manifolds are homotopy equivalent, this homotopy equivalence is a homeomorphism near boundary and $\mathrm{int}(M^{n+1}(G))$ has a metric of (constant) negative sectional curvature.

THEOREM 11 [FJ 2]. Let $f : M_1 \to M_2$ be a homotopy equivalence such that M_1 admits a complete metric of negative curvature and f is a homeomorphism outside of some compact subset of M_1. Then f can be deformed to a homeomorphism and this deformation is constant outside of some compact subset.

THEOREM 12 (F.Waldhausen, W.Thurston). Let $n = 3$, both M_j are compact and $\pi_1(\Omega_j) = 1$. Suppose that M_1 is sufficiently large (for example, $[G,G]/G$ is infinite). Then the group G is quasifuchsian.

REMARK 3. However the question whether or not the manifold $M^4(G)$ is homeomorphic to the product remains open. In this sense the situation is opposite to the case $n = 4$.

EXAMPLE 8 (after, M.Gromov & W. Thurston [GT]). If $n \geq 4$ then there are Kleinian groups $G_n \subset \mathrm{Mob}(\mathbb{S}^n)$ such that: $\Lambda(G)$ is a (tame) topological sphere, $\pi_1(\Omega_j) = 1$ and $M(G)$ is the union of two compact manifolds but the group G_n isn't quasifuchsian.

Commentary. M.Gromov & Thurston [GT] constructed examples of negatively curved compact conformally-flat $n-$manifolds M_n which admit no hyperbolic structure. It's possible to apply their construction to produce examples of *uniformizable* flat conformal manifolds with the same properties. Moreover, $M_n = \Omega_n/G_n$ where G_n are geometrically finite and $\Omega(G_n) = \Omega_n \cup \Omega_n'$ is the union of two simply-connected components. Then $\Lambda(G_n)$ are homeomorphic to $\partial_\infty G_n \cong \mathbb{S}^{n-1}$.

Sketch of construction. Let $\varphi = x_1^2 + ... + x_n^2 - s x_{n+1}^2$ be a quadratic form of the signature $(n,1)$, where s is an algebraic integer such that:

(1) the group Γ_k of automorphisms of φ over $\mathbb{Q}(s)$ is a uniform lattice;

(2) Γ contains a finite cyclic group $\mathbb{Z}_k =< \theta >$ which fixes the plane $\{x : x_1 = x_2 = 0\}$, the even number $k = k(s)$ can be made arbitrarily large.

The group Γ_k is contained in the arithmetic group Γ_k^0 which is the group of automorphisms of the form

$$\varphi_0 = x_0^2 + x_1^2 + ... + x_n^2 - s x_{n+1}^2$$

over $\mathbb{Q}(s)$. Notice that the group Γ_k^0 contains the rotation τ of order 4 which fixes the plane $\{x : x_0 = x_1 = 0\}$. Let L_1 be the hyperplane in the hyperbolic n-space, corresponding to $\{x : x_0 = x_1 = 0\}$. The stabilizer of this hyperplane in Γ_k is a subgroup H_1 which acts as a lattice there. Now the existence of the rotation τ implies that the *normal injectivity radius* of L_1/H_1 in $\mathbb{H}^n/\Gamma_k$ is not less than μ_{n+1} where μ_{n+1} is the *Margulis constant* for $\mathbb{H}^{n+1}$.

Then we can repeat the construction of [GT] for sufficiently large k.

Let $\Delta_k = Ker(\Gamma_k \to \mathbb{Z}_k)$ The orbifold $\mathbb{H}^n/\Delta_k$ splits into k pieces if we remove from it the orbit of L_1/F_1 under the action of $\mathbb{Z}_k$. The closure of each piece is a orbifold U_j whose boundary is totally geodesic outside the closed $n-2$-dimensional totally geodesic suborbifold $N = (L^0 = \{x : x_1 = x_2 = 0\} \cap \mathbb{H}^n)/J$ where J is the stabilizer of L^0 in Γ_k (which acts as an arithmetic lattice on L^0). Denote by E_j the subgroup of Δ_k corresponding to U_j ; $F_{j-1,j}, F_{j,j+1}$ are the subgroups corresponding to the totally geodesic components of $\partial U_j - N$. Then remove one piece U_k from $\mathbb{H}^n/\Delta_k$ and glue the rest to the orbifold O_k identifying the boundary components of $\mathbb{H}^n/\Delta_k - U_k$ via θ. This orbifold admits a metric of negative curvature [GT]; and O_k admits an action of $\mathbb{Z}_{k-1} =< \psi >$ which fixes N. Denote by $\hat{O}_k$ the factor-orbifold by this action.

Our purpose is to construct (for large k) a faithful representation $\rho : \pi_1(\hat{O}_k) \to SO(1+n,1)$, where $G_n = \rho(\pi_1(O_k))$ is a group with the properties above. We choose the unique representation ρ such that:

restrictions of ρ to the subgroups E_j are induced by conjugations in $SO(n+1,1)$.

Then each group $\rho(E_j)$ has an invariant hyperplane V_j in $\mathbb{H}^{n+1}$ so that the angles between V_j and V_{j+1} ($j \in \mathbb{Z}_{k-1}$) tend to π as $k \to \infty$. This fact and the fact that the injectivity radii of boundary components of $\partial U_j - N$ are bounded away from 0 imply that we can apply 2-nd Maskit Combination Theorem to the representation ρ: $\rho(\pi_1(\hat{O}_k))$ is the HNN extension of ρE_2 via the element ψ of finite order (ψ conjugates $\rho(F_{1,2})$ and $\rho(F_{2,3})$).

These examples have another pathological topological property. Let H_n denote the conformal extension of G_n in $\mathbf{Mob}(\mathbb{S}^{n+1})$, $n \geq 5$. Then [FJ 2] implies that all $N_n = \Omega(H_n)/H_n$ are homeomorphic to $\mathbb{S}^1 \times M_n$. Now fix the dimension n and put $M = M_n$. The manifold $N = N_n$ has a flat conformal structure K_0 uniformized by the group $H = H_n$. Apply k times *grafting* along M (see [Go 3]) to the manifold (N, K_0) to produce new flat conformal manifold (N, K_k) $(k = 0, 1, 2, ...)$.

REMARK 4. It's not clear at all: are the smooth manifolds (N, K_0) and (N, K_k) diffeomorphic or not; apriori we have just a quasiconformal equivalence.

ASSERTION. For different k, r the manifolds (N, K_r) and (N, K_k) lie in different connected components of the moduli space $\mathcal{M}(N)$ of all flat conformal structures on N. So, $\mathcal{M}(N)$ consists of infinitely many connected components.

Sketch of proof. The basic idea is to extend topological considerations of [Ka 5] to higher dimensions (I don't know how one can use Riemannian invariants in this case). We need

Lemma 1. Let $H' \subset \mathbf{Mob}(\mathbb{S}^{n+1})$ be a Kleinian group isomorphic to H such that Ω_0/H' is homeomorphic to N for some $\Omega_0 \subset \Omega(H')$. Then H' is topologically conjugate to H in $\mathbb{S}^{n+1}$.

PROOF. Consider the manifolds $R' = (\mathbb{H}^{n+2} \cup \Omega_0)/H'$ and $R = (\mathbb{H}^{n+2} \cup \Omega(H))/H$. These manifolds are homotopy equivalent and this homotopy equivalence extends the homeomorphism between N and $(\Omega_0)/H'$. The manifold R is compact, so R' is compact too. Therefore, $\Omega_0 = \Omega(H')$, H' is geometrically finite and doesn't contain parabolic elements. Thus, applying Theorem 1 , condition of Lemma 1 and Hopficity of H we construct the desired homeomorphism. **QED.**

Then we can repeat [Ka 5]. Suppose that K_t is a path of flat conformal structures on N connecting (say) K_0 with K_1 ; H_t is a corresponding loop of the holonomy groups ($t \in [0,1]$). Since M has no hyperbolic structure all the groups H_t have trivial centralizers in $\mathbf{Mob}(\mathbb{S}^{n+1})$. Therefore, the center of $\pi_1(N)$ is contained in kernels of all holonomy representations and we can consider H_t as images of homomorphisms $\rho_t : H \to H_t$.

Lemma 2. The set E of all t such that H_t is topologically conjugate to H is open and closed in $[0,1]$.

PROOF. The set E is open in $[0,1]$ since the groups H_t being being convex-cocompact are topologically stable [Lok]. Now let H_{t_s}, $\{E \ni t_s\} \to t_\infty$ be any sequence. The group H_{t_∞} is discrete and isomorphic to H; moreover, it's the holonomy group of a flat conformal structure K_{t_∞} on N. Structures K_{t_s} have nonsurjective developing maps [Ka 5], therefore the same is true for their limit [Ka 5]. Hence (see [GK], [Kam], [Ku P 1]) $\Omega(H_{t_\infty})$ contains a component Ω_0 such that: Ω_0/H_{t_∞} is homeomorphic to N. Then we can apply Lemma 1 to finish the proof. **QED.**

Lemma 2 implies that the "grafting number" is a continuous integer invariant along $E = [0,1]$, therefore $K_0 = K_1$. This contradiction finishes the proof of Assertion. **QED**.

End of commentary

Return to the discussion of Theorem 12. If $n = 3$ and M_j aren't sufficiently large then we can't guarantee that G is quasifuchsian (it's true just modulio Thurston's Geometrization Conjecture). However: if M_j is compact and $\pi_1(\Omega_j) = 1$ then $\Omega_j \cup \Lambda(G)$ is homeomorphic to the closed ball (this is a corollary of the result due to M.Bestvina, G.Mess [BM]).

EXAMPLE 9 (B.Apanasov, A.Tetenov [AT 2]).

If we do not assume that $\pi_1(\Omega_j) = 1$ then $\Lambda(G)$ can be a wild sphere in $\mathbb{S}^3$(i.e. there is no a homeomorphism of $\mathbb{S}^3$ which maps $\Lambda(G)$ to a round sphere).

2.3. Now consider the another class of Kleinian groups: $\Lambda(G)$ is a topological circle in $\mathbb{S}^3$.

We know that $\mathbb{S}^1$ can be embedded in $\mathbb{S}^3$ in the following ways:

(a) $\mathbb{S}^1$ is a unknot , i.e there exists a homeomorphism of $\mathbb{S}^3$ which maps $\mathbb{S}^1$ to a round circle.

(b) A nontrivial tame knot: embedding of $\mathbb{S}^1$ is topologically equivalent to polygonal one and (a) doesn't hold.

(c) A wild knot with finitely generated fundamental group of its complement.

(d) A wild knot whose complement has infinitely generated fundamental group.

Consider now which cases can be realized via Kleinian groups.

(a) This is certainly possible.

EXAMPLE 10. Consider a (codimension 1) Fuchsian group $G \subset \mathrm{Mob}(\mathbb{S}^2)$ and extend it to $\mathrm{Mob}(\mathbb{S}^3)$. In this case the limit set of the extended Kleinian group G' is a round circle. Then (if one wishes) we can apply a variation of G' in $\mathrm{Mob}(\mathbb{S}^3)$ to obtain a group whose limit set isn't smooth but satisfies the condition (a).

DEFINITION 1. If a Kleinian group $H \subset \mathrm{Mob}(\mathbb{S}^3)$ is topologically conjugate to a (codimension 2) Fuchsian group, then H is called to be a (codimension 2) quasi-fuchsian group.

(b-c) If for a Kleinian group $G \subset \mathrm{Mob}(\mathbb{S}^3)$ the fundamental group of $\mathbb{S}^3 - \Lambda(G)$ is finitely generated then either it's trivial or cyclic (it's a particular case of Jaco-Hempel's Theorem [H], see also [Kul 2]).

However we still have the possibility:

$\mathbb{S}^3 - \Lambda(G)$ is cyclic and $\Lambda(G)$ is a wild knot.

THEOREM 13 (cf. [Ka 1]). Suppose that G is finitely generated and $\pi_1(\mathbb{S}^3 - \Lambda(G))$ is cyclic. Then $\Lambda(G)$ is a topological unknot and the group G is geometrically finite.

The key point in the proof is that, under the conditions above, $M(G)$ is a Seifert manifold. Thus, it admits an $\mathbb{S}^1$ action that can be lifted in the discontinuity domain and, then, extended to the limit set of our group. The classification due to F.Raymond of topological $\mathbb{S}^1$ actions on $\mathbb{S}^3$ implies that $\Lambda(G)$ is a tame unknot. Therefore, the group G acts as a "convergence group" on $\mathbb{S}^1$, hence this action is conjugate to a Fuchsian action, which is geometrically finite. However, the geometrical finiteness is completely characterized in terms of the topological dynamics on the limit set (see Introduction). Therefore, the group G is geometrically finite.

EXAMPLE 11 [GLT], [Ku], [Ka 2, 3]. There are Kleinian groups $G \subset \mathrm{Mob}(\mathbb{S}^3)$ such that $\Lambda(G)$ are unknotted but G aren't (codimension 2) quasifuchsian groups.

The reason is quite simple: if G is quasifuchsian then $M(G) = \Omega(G)/G$ is a Seifert manifold with the trivial Euler number. However there are examples (see above) when $M(G)$ is a nontrivial circle bundle over an orientable surface.

(d) EXAMPLE 12 (B.Apanasov [A 1], B.Maskit [Mk 1]). There are Kleinian groups $G \subset \mathbf{Mob}(\mathbb{S}^3)$ such that $\Lambda(G)$ are wild knots (with infinitely generated fundamental groups of $\Omega(G)$).

The idea of this example:

Take a knotted necklace of tangent round spheres $\Sigma_1, ..., \Sigma_n$ in $\mathbb{S}^3$. Let G be the group generated by reflections in Σ_j $(j = 1, ..., n)$. Then G is the desired group. To show that $\Lambda(G)$ is indeed a wild knot one should

(i) apply Theorem 1 to prove that $\Lambda(G)$ is homeomorphic to the limit set of some Fuchsian group, and hence it's homeomorphic to $\mathbb{S}^1$;

(ii) apply (for example) Seifert- Van-Kampen Theorem to calculate $\pi_1(\Omega(G))$.

2.4. **REMARK 5**. Quite little is known about Kleinian groups whose limit sets are k-dimensional spheres of codimension > 1 $(k > 1)$. The following interesting results are obtained by I.Belegradek [Bel]:

THEOREM 14. There exist geometrically finite Kleinian groups G_1, $G_2 \subset \mathbf{Mob}(\mathbb{S}^4)$ such that:

1) $\Lambda(G_j)$ is a topological 2-sphere;

2) $\Lambda(G_1)$ is wild;

3) $\Lambda(G_2)$ is tame but the group G_2 isn't topologically conjugate to a codimension 2 Fuchsian group. $\Omega(G_2)/G_2$ is a nontrivial circle bundle over a hyperbolic 3-manifold.

See also the paper of B.Apanasov [A 3].

3. Failure of Ahlfors-Sullivan finiteness theorem in higher dimensions

THEOREM 15. (L.Ahlfors [Ah 1], L.Greenberg [Gre 1], D.Sullivan [Su 1], P.Scott [Sc 1], M.Feighn & G.Mess [FM])
Let G be a discrete non-elementary finitely generated subgroup of $\mathbf{PSL}_2(\mathbb{C})$. Then the factor space $\Omega(G)/G$ consists of a finite number of Riemannian orbifolds

$$S_1, \dots , S_n$$

each having a finite hyperbolic area. In particular, the groups $\pi_1(S_j)$ are finitely generated ($j = 1, \dots , n$) and for every component $\Omega_0 \subset \Omega(G)$ the limit set of its stabilizer G_0 in the group G is the same as $\partial\Omega_0$. Moreover, G has only finitely many cusps and conjugacy classes of finite order elements. The group G is finitely presentable and $M^3(G)$ is homotopy equivalent to a compact orbifold. And, finally, the action of G on $\Lambda(G)$ is recurrent with respect to the Lebesgue measure (i.e. for every measurable subset $A \subset \Lambda(G)$ either $mes(A) = 0$ or $mes(A \cap g(A)) \neq 0$ for some $g \in G - \{1\}$).

Notice, that this Theorem has assertions of the different nature:
(a) topological and algebraic , (b) dynamical: about the limit sets of stabilizers and the measurable dynamics of G on its limit set.

Dynamical statements roughly mean that G has no *wandering domains* in $\Omega(G)$ and *measurable wandering domains* in $\Lambda(G)$.

Now (almost) all topological and algebraic statements have algebraic and topological proofs (see [Ku S], [F Mc], [F M], [M S] and Example 13 below), meanwhile dynamical statements (to the current time) can be proved only analytically. (See, however, Theorem 19 in the section 5).

During some time it was a hope that Theorem analogous to 15 can be proven for Kleinian groups in higher dimensions; an attempt to develop an analytical technique for this aim was made in [Ah 2].

However, it turns out that *all algebraic and topological statements of Theorem 15 fail in higher dimensions. Meanwhile the question about dynamical statements remains open.*

THEOREM 16. [KP 1], [KP 2], [Ka 6], [P 1], [P 2], [P 3]. There exist finitely generated, discrete groups

$F_1, F_2, F_3, F_4 \subset \mathbf{Mob}(\mathbb{S}^3)$ with invariant components Ω_j of their domains of discontinuity ($j = 1, 2, 3, 4$) such that
(i) the groups $\pi_1(\Omega_j/F_j)$ are not finitely generated;
(ii) the group F_1 isn't finitely presentable [P 2] and moreover,
$H_2(F_1, \mathbb{Z})$ has infinite rank;
(iii) F_2 is free and has infinitely many cusps; if $F_{2,n} \subset \mathbf{Mob}(\mathbb{S}^n)$ is the conformal extension of F_2 to $\mathbb{S}^n$ ($n \geq 3$) , then

$$\mathbf{rank}(H_{n-1}(\Omega(F_{2,n})/F_{2,n}, \mathbb{Z})) = \infty \ .$$

(iv) F_3 has infinitely many conjugacy classes of finite order elements [K 6].

(v) F_4 is a normal subgroup of a convex-cocompact group $G_4 \subset \mathrm{Mob}(\mathbb{S}^3)$ and also has infinitely many conjugacy classes of finite order elements,

$G_4/F_4 \cong \mathbb{Z}$ [P 1].

REMARK 6. In particular, the group G_4 is word-hyperbolic.

Moreover, it's possible to construct an example $F_5 \subset \mathrm{Mob}(\mathbb{S}^3)$ such that $\Omega(F_5)/F_5$ has infinitely many connected components.

The manifold $M_1 = \Omega_1/F_1$ is homeomorphic to the double (along a part of boundary) of a cube with infinitely many handles.

The manifold M_2 is homeomorphic to an open handlebody X_r from which some ∞-component link L is removed. Each component of the link is an unknot presenting free generator of $\pi_1(X_r)$.

The group F_2 is interesting also from the algebraic point of view: small deformations of it produce a family of finitely generated discrete subgroups F_3 of $\mathbf{SO(4\ ,\ 1)}$ each possessing infinitely many conjugacy classes of finite order elements.

THEOREM 17 [Ka 6]. For each $q \in \mathbb{Z}$ there exist :

1) an integer r =(rank of free group $G_r =< x_1, \dots, x_r >$),

2) automorphism $\varphi : G_r \to G_r$ such that the group

$$\Gamma(r, q) =< x_1, \dots, x_r : (\varphi^n(x_1))^q = 1, n = 1, 2, \dots >$$

has infinitely many conjugacy classes of finite order elements $[y_n = \varphi^n(x_1)]$ and $\Gamma(r, q)$ admits discrete faithful representation $\rho : \Gamma(r, q) \to \mathbf{Mob}(\mathbb{S}^3)$.

REMARK 7. Suppose that $\Gamma \subset \mathrm{GL}(n, \mathbb{R})$ is a finitely generated group which contains infinitely many conjugacy classes of finite order elements. Then, according to Selberg's lemma [Se] Γ has a torsion-free finite index normal subgroup Γ_0. Then the arguments of [F M] show that the group Γ_0 can't be the fundamental group of a compact aspherical complex. Really, in other case:

(a) $M_0 = \Gamma_0 \backslash \mathrm{GL}(n, \mathbb{R})/SO(n)$ has a finite homotopy type ;

(b) the fixed-point set for action on M_0 of any element k of order $q \in K = \Gamma/\Gamma_0$ can have not more than

$$\nu_k = \sum_j rank(H_j(M_0, \mathbb{Z}/q))$$

connected components;

(c) therefore, the number of stratums of the singular set of the orbifold M_0/K is bounded by $|K|\max\{\nu_k\}$.

The underlying space of the orbifold $\Omega(r, q)/\rho\Gamma(r, q)$ is homeomorphic to X_r , where the singular set is L (see above). Generalizations and discussions of Theorems 16 , 17 see in [P1], [P3], [Bo M].

Suppose that G is a countable group of homeomorphisms of the closed 3-ball $\mathbb{B}^3$. Then the discontinuity domain for this action can be defined as a *maximal* open set $D \subset \mathbb{B}^3$ such that G acts properly on D (i.e. for every compact set $K \subset D$ there is only finitely many $g \in G$ such that $gK \cap K \neq \emptyset$).

EXAMPLE 13. The infinite cyclic group $G = \mathbb{Z}$ admits two actions ϕ_1, ϕ_2 on $\mathbb{B}^3$ such that:

(i) this actions are proper on the interior of $\mathbb{B}^3$;
Denote by $D_j \subset \mathbb{S}^2$ discontinuity domains for these actions ($j = 1, 2$).
(ii) $D_1/\phi_1(G)$ contains infinitely many simply connected components;
(iii) $D_2/\phi_2(G)$ contains infinitely many annuli.

According to [KS] this is the only possible topological deviation (from the Ahlfors' finiteness theorem) for proper actions of finitely generated groups G on the interior of $\mathbb{B}^3$ (in particular, for every such action the part S^- of $(D \cap \mathbb{S}^2)/G$ which consists of components with negative Euler characteristic have to have a finite topological type).

The idea of the examples is rather simple. First, start with the action ϕ_0 of $\mathbb{Z} =< t >$ on $\overline{\mathbb{R}}^3_+$:

$t : (x_1, x_2, x_3) \mapsto (2x_1, \frac{1}{2}x_2, x_3)$, where $x_3 \geq 0$.

A discontinuity domain $D(\phi_0)$ for the restriction of this action to $\overline{\mathbb{R}}^2$ is $\mathbb{R}^2 \backslash \mathbb{R}^1$. D_0/G consists of two annular components. Identify $\mathbb{R}^2$ with $\mathbb{C}$ and take pull back for this action under the map $z \mapsto z^2$. Denote the new action by ψ_1, then $D(\psi_1)$ and $D(\psi_1)/\psi_1(G)$ consist of 4 components. Iterating this process we can obtain arbitrarily large (but finite) number of annular components of the factor-surface. Thus we have to pass correctly to a limit.

To obtain infinite number take a Cantor set $K \subset \mathbb{S}^1 \subset \mathbb{C}$ and the cone $C(K) = \{z \in \mathbb{C} : \lambda z \in K \text{ for some } \lambda \in \mathbb{R}\}$. Then "glue" to each component of $\mathbb{C} - C(K)$ a copy of the action ϕ_0 on the upper half-plane. Extend this action to $\mathbb{R}^3$ via the trivial action on x_3. The result is the action ϕ_2.

To obtain ϕ_1 one can argue in the following way. Take the interval $[1, 2]$ on the horizontal line $\{x_2 = 0\}$ in $\mathbb{R}^2$. Let Q be any Cantor set on $[1, 2]$. Then "blow up" our action ϕ_0 along all complementary intervals I_k to Q in $[1, 2]$ inserting discs instead of I_k. As the result we will obtain infinitely many equivalence classes of simply connected wandering domains in the discontinuity set of the new action ϕ_1.

Idea of proof of Theorem 16.

Probably the simplest way to explain the idea of construction of the group F_2 is the following.

Suppose that we have a hyperbolic 3-manifold M^3 of finite volume which is fibered over $\mathbb{S}^1$ with the fiber Σ. Then

$$1 \to \pi_1(\Sigma) \to \pi_1(M^3) \to \mathbb{Z} =< t > \to 1$$

and the action of t by an automorphism τ on $\pi_1(\Sigma)$ is induced by a pseudo-Anosov homeomorphism of Σ. Then, take a nontrivial primitive loop γ on Σ which isn't parallel to any cusp. Imagine that we constructed a faithful discrete representation

$$\rho : \pi_1(M^3) \to SO(4, 1)$$

with the image H_2 such as: $\rho(\gamma)$ is a parabolic element. Then the group $F_2 = \rho\pi_1(\Sigma)$ has the desired properties. The infinite number of cusps comes from the sequence of maximal parabolic subgroups $< \rho(\tau^k(\gamma)) >, k \in \mathbb{Z}$ in the group F_2. These groups are nonconjugate one to another since pseudo-Anosov homeomorphism is irreducible. Originally, instead of pseudo-Anosov homeomorphisms we used reducible ones [Ka 6]. Later, the original idea about pseudo-Anosovs was realized by L.Potyagailo [P 3].

It's rather easy to construct a representation ρ with the properties above if instead of $SO(4,1)$ we take the isometry group of some negatively curved 4-manifold (compare [Gr 1]). From geometrical point of view we need to take $[0,1] \times M^3$ and "open cusp" along γ on $\{1\} \times M^3$ so that the resulting manifold is complete, negatively curved and has convex boundary.

To construct (in the category of negatively curved manifolds) Example of the group F_3 one need not only to open the cusps, but also "attach a branched 2-handle" $H(q)$ to $\{1\} \times M^3$ along γ so that in the fundamental group of the orbifold $[0,1] \times M^3 \cup H(q)$ the element γ has the order q. The simplest way how one can construct example of finitely generated Kleinian group which isn't finitely presentable appears in [Bo M]. Suppose that the hyperbolic manifold M^3 as above contains a totally geodesic surface Ψ of finite area. Let N^3 be the covering of M^3 constructed by the subgroup $\pi_1(\Sigma)$. Denote by Φ the a component of lift of Ψ in N^3. Then the fundamental group of Φ isn't finitely generated. Now, take two copies of N^3 and identify them along Φ. The resulting complex has negative curvature and one can realize it as a deformation retract of a complete negatively curved manifold P^4 (cf. [G T]). Then the fundamental group of P^4 is finitely generated (by generators of two copies of $\pi_1(N^3)$) . But $\pi_1(P^4)$ is not finitely presentable since it's amalgamated free product along free group of infinite rank. This construction appears as a simplification of [KP 1], where the amalgamated subgroup was a normal subgroup in a Schottky-type group.

So, the real problem is to find example with constant negative curvature. However it's rather technical than conceptual question. From the point of view of [Gr 1] and [GT] we use the extended residual finiteness for transition to some finite-index subgroup where the "normal injectivity radius" of γ and/or Φ is sufficiently large.

4. Deformations of Kleinian groups

By a *small deformation* of a subgroup $G \in \mathbf{Mob}(\mathbb{S}^n)$ we shall mean an element of some small neighborhood of $[id]$ in the variety

$$R(G,n) = Hom(G, SO(n+1,1))/SO(n+1,1).$$

As we have seen in the section 2, one can easily construct deformations of quasi-fuchsian groups in $\mathbf{Mob}(\mathbb{S}^2)$.

4.1. Small deformations of (codimension $n-1$) quasifuchsian groups in higher dimensions.

Let Γ be a finitely generated subgroup of a reductive Lie group $\mathbf{G}$ of dimension d over $\mathbb{R}$ (for example, $\mathbf{G} = \mathrm{SO(n,\ 1)}$, $d = n(n+1)/2$). Denote by $\mathcal{G}$ the Lie algebra of $\mathbf{G}$. Then we can consider $\mathcal{G}$ as a Γ-modulus with respect to the adjoint representation. We shall consider the cohomology $H^1(\Gamma, \mathcal{G})$ as the space of *infinitesimal deformations* of the group Γ in $\mathbf{G}$.

Suppose now that Γ is isomorphic to a cocompact lattice in $PSL(2,\mathbb{R})$. Then

$$\Gamma = < a_1, b_1, ..., a_g, b_g, c_1, ..., c_k :$$

$[a_1, b_1] \cdot ...[a_g, b_g] \cdot c_1 \cdot ... \cdot c_k = 1, c_j^{r_j} = 1, j = 1, ..., k >$.

Denote by e_j the number

$$codim_{\mathcal{G}}\{\xi \in \mathcal{G} : Ad(c_j)(\xi) = \xi\} = e_j$$

($d-$ dimension of centralizer of $< c_j >$ in $\mathbf{G}$) .

Let s denote the dimension of centralizer of Γ in $\mathbf{G}$. Then (A.Weil [W]):

$$h = dimH^1(\Gamma, \mathcal{G}) = -2d + 2s + e_1 + ... + e_k \qquad (*)$$

Moreover, if $s = 0$ then $Hom(\Gamma, \mathbf{G})/\mathbf{G}$ is a smooth manifold of dimension $dimH^1(\Gamma, \mathcal{G})$ near the point $[id]$ (see also [Si]).

As example consider the case of "triangle groups". Let $\Gamma \subset PSL(2, \mathbb{R})$ be a group with the presentation

$$\Delta(r_1, r_2, r_3) =< c_1, c_2, c_3 : c_1 \cdot c_2 \cdot c_3 = 1, c_j^{r_j} = 1, j = 1, 2, 3 >$$

where $r_1^{-1}+r_2^{-1}+r_3^{-1} < 1$. This group always exists, discrete and $\Lambda(\Gamma) = \mathbb{S}^1$. These groups are called to be hyperbolic triangle groups. Such groups are locally rigid in $SO(3, 1)$ (as it follows, for example, from the vanishing of $H^1(\Gamma, so(3, 1))$). Moreover, triangle groups are "superrigid" in $PSL(2, \mathbb{C})$, i.e. every representation of Γ in $PSL(2, \mathbb{C})$ either extends to a representation of $PSL(2, \mathbb{R})$ or is conjugate either to $SO(3)$ or the group of upper-triangular matrices in $PSL(2, \mathbb{C})$ (see L.Greenberg [Gre 2]).

However, the situation changes if we consider $\mathbf{G}= SO(4, 1)$ instead of $PSL(2, \mathbb{C})$. First let $\Gamma_1 \subset \mathbf{G}$ be the group obtained by "trivial" extension $SO(2, 1) \subset SO(4, 1)$. Then $d = 10, s = 1, e_j = 6$ $(j = 1, 2, 3)$. So, (*) tells us that Γ_1 is locally rigid in $\mathbf{G}$. However, instead of the "trivial extension" Γ_1 we can take a nontrivial one. Suppose that there are some numbers $k_j \in \mathbb{Z}, |k_j| > 1$ $(j = 1, 2, 3)$ such that:

$k_1^{-1} + k_2^{-1} + k_3^{-1} = 0$,

k_j divide r_j $(j = 1, 2, 3)$.

Then embed the hyperbolic plane $\mathbb{H}^2$ as a geodesic plane in $\mathbb{H}^4$. Take rotations θ_j around $\mathbb{H}^2$ to the angles $2\pi/k_1, 2\pi/k_2, 2\pi/k_3$ in the hyperbolic 4-space. Then define $\tilde{c_j} = \theta_j \circ c_j$ where c_j are realized as generators of Γ_1.

Put Γ_2 = group generated by $\tilde{c_j}, j = 1, 2, 3$. It's easy to see that Γ_2 is a (codimension 2) quasifuchsian subgroup of **Mob** $(\mathbb{S}^3)$. If $r_j > 3$ for every j, then $e_j = 8$ and the formula (*) gives the dimension h= 4 for $H^1(\Gamma_2, \mathcal{G})$. In the same way one can realize even 6 as the dimension of the cohomology group.

However, the question which arises immediately is: do the infinitesimal deformations above tangent to curves in the $R(\Gamma_2, 3)$? It's easy to show that $dimH^2(\Gamma_2, \mathcal{G}) = 1$, so in principle we can have some obstructions here. As it follows from [GM] the only obstruction that we should care about is the "cup-product"

$$[\ ,\ ] : H^1(\Gamma_2, \mathcal{G}) \times H^1(\Gamma_2, \mathcal{G}) \to H^2(\Gamma_2, \mathcal{G}) \cong \mathbb{R}$$

On other hand, any irreducible representation ρ of the group $\Delta(2, 3, p)$ in $SO(4, 1)$ has zero-dimensional group

$$H^1(\Gamma, \mathcal{G}_{Ad\circ\rho}) \quad .$$

So, the group $\Delta(2,3,p)$ has only finitely many conjugacy classes of representations in $SO(4,1)$. There are no Zariski dense representations of $\Delta(2,3,p)$ in $SO(4,1)$ (N.Kuiper, P.Waterman). In particular this means that any discrete and faithful representation of $\Delta(2,3,p)$ has invariant hyperbolic plane in $\mathbb{H}^4$.

4.2. However, if G has codimension $< n-1$ then the existence of any deformations is much more questionable. Suppose that G is a (codimension 1) Fuchsian group in $\mathbf{Mob}(\mathbb{S}^n)$ such that $\mathbb{H}^n/G$ is compact. Then one has the following sources of deformations:

(a) Bending along closed totally geodesic hypersurface (see [G 2], [JM], [K]).

(b) Some algebraic machinery to create "infinitesimal" deformations of G in the arithmetic case (like [LM]).

(c) Simple geometric construction of "infinitesimal" deformations of G if G is a reflection group [Ka 7]. In this case one can show that

$$dim H^1(G, so(n+1,1)_{Ad}) = f - n - 1$$

where f is the number of "faces" of the fundamental polyhedron for G.

(d) Examples of infinitesimal deformations which can't be extended to small deformations since have nontrivial 1-st obstruction [JM].

(e) Generalized "bending" applied not to a graph of groups with (codimension 1) Fuchsian edge subgroups, but rather to a complex of groups (the first examples of such opportunity were discovered in [GT] and by S.Tan), see [AT 3], [KM]. One of rather strange particular cases is the following. Suppose that $M = M(G)$ contains an *immersed* totally geodesic hypersurface $f : \Sigma \to M$ such that: (1) each point can have not more than 2 preimages;

(2) if S is the set of "selfintersections" of $f(\Sigma)$ then each component of $f^{-1}S$ splits Σ.

Then $H^1(G, so(n+1,1)_{Ad}) \neq 0$.

And nothing more to this date if $n > 3$

(f) If $n = 3$ then we have also example of "Stumping deformations" [A 2], generalized by [Tan].

(g) The first obstruction for infinitesimal deformations is always trivial (J.Millson).

(h) In the case of a reflection group G we know that space of small deformations of G is smooth and has dimension $f - 4$.

(i) An example of "stamping along isolated geodesic" (B.Apanasov [A 2]).

(j) Examples of G which have no even infinitesimal deformations: given a 2-bridge knot K in $\mathbb{S}^3$ one can find infinitely many Dehn surgeries on K such that:

$$H^1(G, so(4,1)_{Ad}) = 0$$

where G is the fundamental group of hyperbolic manifold obtained by such Dehn surgery [Ka 7]. So, G has no even infinitesimal deformations.

And again, nothing more but the following

CONJECTURE 1 [Ka 2, Ka 7]. The group G admits a nontrivial small deformation in $SO(4,1)$ iff the orbifold $\mathbb{H}^3/G$ contains an embedded incompressible sub-orbifold $\mathcal{O}$ such that $\pi_1(\mathcal{O})$ has trivial normalizer in G.

4.3. **"Large deformations"**. It's well known that the limit ρ_∞ of a sequence of discrete faithful representations ρ_k in $\mathbf{Mob}(\mathbb{S}^n)$ of any finitely generated group

G is again discrete and faithful (see for instance, [J], [Mart 1]). One other hand, if $G \subset Isom(\mathbb{H}^n)$ is a lattice, then any such sequence is precompact in $R(G, N)$ provided that $n > 2$ (J.Morgan [Mo 2]). Moreover, this sequence is precompact even if one substitutes the condition "discrete and faithful" by: G has no nontrivial decomposition in an amalgamated free product (see also [Mo 1], [Th 1]). So, for $n = 3$ this agrees with the Conjecture above. However, there are examples of reflection groups such that $dim(R(G, 3)) > 0$, G splits in an amalgamated free product but $R(G, 3)$ is compact.

Denote by $AH(G, N)$ the space of conjugacy classes of discrete and faithful representations of a group G in $\mathbf{Mob}(\mathbb{S}^N)$.

CONJECTURE 2 There exists a function $\phi(n, N, v)$ such that: for any lattice $G \subset Isom(\mathbb{H}^n)$ $(n > 2)$, and for any $\rho \in AH(G, N)$ there exist a point $x \in \mathbb{H}^{N+1}$ and a system of generators $\{g_1, ..., g_k\}$ of the group G such that:

$$d(x, \rho(g_j)) \leq \phi(n, N, Vol(\mathbb{H}^n/G = M(G)))$$

4.4. **"Pinching vs. Pinching"**.

Let $G \subset \mathbf{Mob}(\mathbb{S}^n)$. An element $g \in G$ is "pinched" by a representation ρ lying on the boundary of the space $AH(G, n)$ if g is loxodromic, but $\rho(g)$ is parabolic. Suppose that $n = 2$, G is a torsion-free Fuchsian group such that $S = \mathbb{H}^2/G$ is a closed surface. Then we know that $g \in G$ can be pinched iff g is represented by some simple closed loop on S. More generally, if $\{g_1, ..., g_k\} \subset G$ is a collection of elements, then it can be pinched iff this set of elements can be split in 2 subsets A, B such that each of them can be represented (up to conjugation in G) by some system of simple pairwise disjoint loops (therefore, $k \leq dim_{\mathbb{R}} T(S)$, where $T(S)$ is the Teichmuller space of S).

Now consider the action of G on $\mathbb{H}^4$, and representations of G in $\mathbf{Mob}(\mathbb{S}^3)$. Then, according to Theorem 16 we can pinch simultaneously an infinite set of elements represented by simple closed loops on S. On other hand, it easy to construct examples of pinching of *nonsimple* loops on S. Moreover, given any finite set L of closed loops on S one can construct a negatively pinched 4-manifold M^4 with the fundamental group G such that for any element of L can't be represented by closed geodesic in M^4. However, the pinching constants for M^4 will depend on L and it's impossible to find a universal pinching constant a (independent on L) for which such pinching is possible.

CONJECTURE 3. For every finite set $L = \{g_1, ..., g_k\} \subset G$, there exists a number $n = n(G, L)$ such that:

the system of elements L can be pinched in $AH(G, n)$.

REMARK 8. The number n have to depend on the both G and L. There exists a finite number of elements $I = \{h_1, ..., h_m\} \subset G$ such that the variety $E(I) = \{\rho \in Hom(G, SO(n+1, 1) : \text{all elements } \rho(h_j) \text{ are parabolic }\}/SO(n+1, 1)$ is relatively compact in $Hom(G, SO(n+1, 1))/SO(n+1, 1)$. There are two ways to show this.

Arguing algebraically one can find a finite number of elements whose characters determine the conjugacy class of any nonelementary representation. Assumption that their images are parabolic provides the upper bound $2n$ on their traces.

More geometrically, notice that we can take a system of geodesics J which decomposes the surface $\mathbb{H}^2/G$ on pants and take one extra geodesic γ which intersects all curves in J. Then take representatives of G-conjugacy classes of elements of $J \cup \{\gamma\}$. This is the system I. To prove this one can use pleated surfaces.

However, **all** elements of the group G can't be pinched simultaneously, therefore taking an exhausting increasing sequence of finite subsets $G(q)$ such that $I \subset G(q) \subset G$ we have to end up with some finite set L that can't be pinched in the given dimension n.

Nevertheless, the "size" of the subvariety $E(I)$ is increasing as $n \to \infty$. In the limit it "blows up" as the group $SO(n)$ becomes noncompact group $O(\infty)$ of orthogonal automorphisms of L_2, which contains discretely embedded G.

4.5. **Topological constrains on Kleinian groups**

There are several restrictions on topology of Kleinian groups provided by compactness theorems. For example:

4.5.1. Consider a pair of complete hyperbolic 3-manifolds N_j each having one cusp. Consider various 3-manifolds N obtained from N_j by gluing of their ideal boundary components. Then the number of N such that $\pi_1(N)$ admits a discrete faithful representation in $\mathbf{Mob}(\mathbb{S}^n)$ is finite (see the discussion in 4.3).

4.5.2. Suppose that $G \subset \mathbf{Mob}(\mathbb{S}^3)$ is a torsion-free Kleinian group; $\sigma_j : \Sigma_j \to \mathbb{H}^4/G$ $(j = 1, 2)$ be two continuous maps of closed oriented surfaces Σ_j inducing monomorphisms of fundamental groups. Then

$$| < \sigma_1, \sigma_2 > | \leq C(|\chi(\Sigma_1)|, |\chi(\Sigma_2)|)$$

where $< ., . >$ is the algebraic intersection number and C is a universal function of two variables (independent on M) [Ka 8].

In particular, suppose that G_1, G_2 are two purely loxodromic pseudofuchsian groups contained in some torsion-free Kleinian subgroup of $\mathbf{Mob}(\mathbb{S}^3)$. Then the linking number between $\Lambda(G_1)$ and $\Lambda(G_2)$ is bounded from above by some function depending only on the number of generators of G_1, G_2.

4.5.3. Another constrain on parabolic elements in Kleinian groups was discussed in 4.4.

5. Quasiconformal vs. negatively curved (generalizations of Kleinian groups)

There are several ways to perturb the Mobius geometry. In particular:

(i) Consider **quasiconformal** instead of **conformal**;

(ii) Consider **negative curvature** instead of **constant negative curvature**.

Let's explain these ideas.

5.1. **Quasiconformal.** If one look at the group $\mathbf{Mob}(\mathbb{S}^n)$ "from the infinity of the hyperbolic space", then this is just the group of *conformal automorphisms* of $\mathbb{S}^n$. Therefore, it's natural to perturb **conformal** to **quasiconformal.**

Definition 2. The group G acting on $\mathbb{S}^n$ is called to be *uniformly quasiconformal* if there exists a constant $K = K_G$ such that:

the quasiconformal dilatation $K(g)$ is bounded by K for each $g \in G$.

If $n = 2$ then there is no any difference between conformal and uniformly quasiconformal groups (basically, because the Beltrami equation is solvable in this dimension):

THEOREM 17 (P.Tukia [Tu 2]). If G is a uniformly quasiconformal group acting on $\mathbb{S}^2$ then G is quasiconformally conjugate to a conformal action.

If G is a uniformly quasiconformal group acting on $\mathbb{S}^n$ then we shall write: $G \in QC(n)$.

The situation changes dramatically with transition from 2 to 3.

First (nondiscrete) examples of uniformly quasiconformal groups acting on $\mathbb{S}^n$ for which Theorem 17 fails were constructed by P.Tukia [Tu 2]. Examples of discrete groups of such kind which are not even topologically conjugate to Kleinian groups were produced in [FS 1], [Mart 2], [Ka 2]. All these examples appeal to presence of elements of finite order. For example, take the group G from the Example 11, let $\mathbb{Z}_n$ be a finite cyclic group of quasiconformal transformations of $\mathbb{S}^3$ which commute with all elements of G (such group $\mathbb{Z}_n$ can be lifted from the factor-manifold $\Omega(G)/G$). Then the group $\mathbb{Z}_n \times G$ is uniformly quasiconformal but can't be topologically conjugate to a conformal group , since G isn't quasifuchsian.

If a quasiconformal group H has no elements of finite order, then it's not so easy to explain: why H can't be topologically conjugate to a conformal group, even if "it's evident that such conjugation is impossible" (see [FS 1] and [Mart 4]). It seems to be that the first example of such kind is given by [Ka 8]: one have to show that

(a) The total space E of a circle bundle over a surface S whose Euler number is sufficiently large with respect to the genus of S can't be uniformized by any Kleinian group (see [Ka 8]).

(b) The manifold E can be uniformized by a uniformly quasiconformal group (here one can apply the technique of [FS 1]).

5.2. **Negative curvature.** Consider a complete 3-manifold M with negatively pinched sectional curvature. Then the ideal boundary of its universal covering $\tilde{M}$ is homeomorphic to $\mathbb{S}^2$ and the action of $G = \pi_1(M)$ extends to a discrete action on $\mathbb{S}^2$.

It can occur that G has a nonempty discontinuity domain $\Omega(G)$ at infinity and, so we can consider G as a generalized Kleinian group. In such case we shall write: $G \in NP(2)$ (or $G \in NP(n)$ if the sphere at infinity is n-dimensional). Denote by $N(G)$ the *Nielsen's kernel* of G, i.e. the minimal convex subset of M homotopy equivalent to M itself.

THEOREM 18. Suppose that $vol(N(G)) < \infty$. Then G is topologically conjugate to a Mobius action. (This is a corollary of Thurston's geometrization theorem).

"Geometrically infinite case" of finitely generated G isn't clear even modulio the geometrization (analogous question was addressed also in [Mart 4] and [MS]). The problem splits in two components.

(a) Behavior of G on its discontinuity domain. The question that seems to be open is the analogue of Ahlfors finiteness theorem (the dynamical part, since the topological part is proven).

The best that we can prove here (to the current date) is:

THEOREM 19. If G is finitely generated, torsion-free and either $N(G)$ can be compactified to a manifold with boundary or G satisfies the condition (*) of F.Bonahon [Bon]:

(*) "G can't be decomposed in free product so that any parabolic element of G is conjugate to one of free factors"

then the Ahlfors finiteness theorem holds for G.

Idea of the proof. Suppose that some component X of $\partial N(G)$ contains an embedded discs $D(q)$ of arbitrary large size, which is equivalent to the assumption that for some lift $\tilde{X}$ of X in the universal covering has the property:

the limit set of the stabilizer of $\tilde{X}$ in G isn't equal to the ideal boundary of $\tilde{X}$.

Take a sequence of ϵ_q such that the ϵ_q-neighborhood $U(q)$ of $D(q)$ is still embedded in $N(G)$. Then one can consider X as a part of *geometrically infinite end* E and find a sequence of closed geodesics in $N(G)$ that "exit this end". Then the point is that one can reverse the arguments of F.Bonahon [Bon] and R.Canary [Can] with respect to the end E and find a sequence of *ruled simplicial surfaces* F_n which "exit the end E". However, F_n have bounded area and, thus, can't contain embedded discs $F_n \cap D(q)$ of arbitrary large diameter. This contradiction concludes the proof.

(b) Action of G on its limit set. Suppose for a moment that G is torsion-free, and satisfies either to the condition (*) or $N(G)$ can be compactified to a manifold with boundary. Then we have an analogue of Bonahon's theorem [Bon] about geometrical tameness of geometrically infinite ends of M [Can]. Thus, we can define the "ending laminations" for the geometrically infinite ends of M. Suppose, moreover, that the compactification of $N(G)$ is acylindrical. Then, given M as above, one can construct a Kleinian group $F \subset \mathbf{Mob}(\mathbb{S}^2)$ isomorphic to G so that :

(1) the action of F on $\Omega(F)$ is topologically equivalent to the action of G on $\Omega(G)$;

(2) for each geometrically infinite end E of M the ending lamination $\epsilon(E)$ is isomorphic to the ending lamination of a geometrically infinite end E' of $\mathbb{H}^3/F$.

Then the problem is to construct a weak topological equivalence between G and F (sf. [CT] and topological vs quasiconformal in *Introduction*).

So, even in the case of $\mathbb{S}^2$ quasiconformal groups are much closer to the Mobius ones than the fundamental groups of negatively pinched manifolds.

Now discuss the situation with $n = 3$. First, there is an example of $G \in QC(3)$ which has no a proper extension in $\mathbb{H}^4$ [FS 1]. There are examples $G \in NP(3)$ which can't be made quasiconformal. For example, take G to be the integer Heizenberg group acting on the ideal boundary of the complex- hyperbolic space [FS 1]. The basic reason is that G acts properly on $\mathbb{B}^4$ and there is no a negatively pinched G-invariant metric on $\mathbb{B}^4$ which is quasiisometric to $\mathbb{H}^4$.

CONJECTURE 3. Suppose that G is a quasiconformal group acting on $\mathbb{S}^3$ such that the action of G on its limit set is expanding (each limit point is a "point of approximation"). Then G is a hyperbolic group in sense of Gromov [Gr 2] (in particular, G is finitely generated). If G admits a uniformly quasiconformal extension to $\mathbb{H}^4 \cup \mathbb{S}^3$ then $\mathbb{H}^4/G$ admits a metric of negatively pinched sectional curvature.

5.3. One can unify two previous generalizations of Kleinian groups in the class of *convergence* groups (see [Mart 4] and references there).

Definition 3. The group G of homeomorphisms acting on a locally compact space X is called a *convergence* group if for every sequence $\{g_n\}$ in G there exist points $x, y \in X$ and a subsequence $\{g_{n_k}\}$ such that: on $X - \{x\}$ the sequence $\{g_{n_k}\}$ converges[Th 1] uniformly on compacts to the map $f : X - \{x\} \to \{y\}$.

There are examples of a convergence group G acting on $\mathbb{S}^2$ and isomorphic to the fundamental group of a closed surface such that G isn't topologically conjugate to a Kleinian group (see [Mar 3], [Mart 4).

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