Artin groups, projective arrangements and fundamental groups of smooth complex algebraic varieties

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Abstract. We prove that for any affine variety S defined over $\mathbb{Q}$ there exists an Artin group G such that a Zariski open subset U of S is biregular isomorphic to a Zariski open subset U' of the character variety $X(G, PO(3)) = \mathrm{Hom}(G, PO(3))//PO(3)$. The subset U contains all real points of S. As an application we construct new examples of finitely-presented groups which are not fundamental groups of smooth complex algebraic varieties.

Les groupes d'Artin, arrangements projectifs et groupes fondamentales des variétés complexes algébriques lisses

Résumé. Nous montrons que pour toute variété affine S définie sur $\mathbb{Q}$ il existe un groupe d'Artin G tel qu'un sous-ensemble ouvert de Zariski U de S est birégulièrement isomorphe à un sous-ensemble ouvert de Zariski de la variété des classes de représentations $\mathrm{Hom}(G, PO(3))//PO(3)$. Le sous-ensemble U contient tous les points réels de S. Comme application nous construisons de nouveaux exemples de groupes de présentation finie qui ne sont pas le groupe fondamental d'une variété complexe algébrique lisse.

Version française abrégée

Soit A un graphe bipartite, de parts $\mathcal{P}$ et $\mathcal{L}$. Nous disons que p dans $\mathcal{P}$ et ℓ dans $\mathcal{L}$ sont *incidents* si p et ℓ sont joints par une arrête. Une *réalisation* de A (comme arrangement dans $\mathbb{P}^2$) est une application ϕ de $\mathcal{P}$ dans $\mathbb{P}^2$ et de $\mathcal{L}$ dans l'espace projectif dual $(\mathbb{P}^2)^\vee$ des droites de $\mathbb{P}^2$, qui respecte l'incidence. Précisant un resultat de Mnev [6], nous montrons que pour tout schéma affine S de type fini sur $Spec(\mathbb{Z})$, il existe A tel que S soit un ouvert de Zariski d'un espace de modules de réalisations de A. A certains graphes bipartites A, suffisant pour réaliser tout schéma affine, nous attachons un groupe d'Artin (resp. de Shephard) correspondant G_A^a (resp. G_A^s), tel qu'un ouvert et fermé de Zariski, $\mathrm{Hom}_f^+(G_A^a, PO(3))//PO(3)$ de l'espace $\mathrm{Hom}(G_A^a, PO(3))//PO(3)$ soit, sur $\mathbb{Q}$, isomorphe à un sous-espace ouvert de Zariski W de l'espace de modules des réalisations de A. De plus, W contient tous les points réels. Les représentations dans $\mathrm{Hom}_f^+(G_A^a, PO(3))$ se factorisent par G_A^s. Nous en déduisons que les schémas des classes de représentations de groupes d'Artin et de Shephard dans $PO(3)$ peuvent avoir des singularités arbitrairement compliquées (même en des points correspondant à une représentation d'image finie).

Par contre si un groupe est le groupe fondamental d'une variété complexe algébrique lisse nous avons des rèstrictions sévères sur ces singularités selon le théorème suivant qui est un conséquence d'un théorème de Hain [2]:

Théorème. *Soit M une variété complexe algébrique lisse, G un groupe reductif réel et $\rho : \pi_1(M) \to G$ une représentation d'image finie. Alors le germe (analytique réel) $(\mathrm{Hom}(\pi_1(M), G), \rho)$ et son complexifié sont des cones quasi-homogènes avec des générateurs de poids 1 et 2 et des relations de poids 2, 3 et 4. Supposons en plus que les orbites de $Ad(G)$ dans $\mathrm{Hom}(\pi_1(M), G)$ admettent une section locale qui passe par ρ. Alors le germe quotient $(\mathrm{Hom}(\pi_1(M), G)//G, [\rho])$ et son complexifie sont des cones quasi-homogènes avec des générateurs de poids 1 et 2 et des relations de poids 2, 3 et 4.*

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1. Introduction. What follows is an announcement and short account of the contents of [4]. Our work concerns Serre's problem of determining which finitely-presented groups are fundamental groups of smooth complex algebraic varieties. The first examples of finitely-presented groups which were not fundamental groups of (not necessarily compact) smooth complex algebraic varieties were given by Morgan [7]. We find a new class of such examples which consists of Artin and Shephard groups. Since all Artin and Shephard groups have quadratically presented Malcev algebras (see [4]), Morgan's test does not suffice for this class of groups. This result is surprising because the basic examples of Artin groups, e.g. free groups, free Abelian groups, braid groups and, more generally, Artin groups corresponding to finite Coxeter groups, are fundamental groups of smooth complex quasi-projective varieties.

Our work on Serre's problem follows from combination of theorems 1 and 2 below:

Theorem 1. *For any affine variety S over $\mathbb{Q}$ (not necessarily reduced or irreducible) there are Shephard and Artin groups G such that a Zariski open subset U of S is biregular isomorphic to a Zariski open subset U' in the character variety $X(G, PO(3)) = \mathrm{Hom}(G, PO(3))//PO(3)$. The subset U contains all real points of S.*

We combine Theorem 1 with the following theorem that we deduce from Theorem 7 below and a theorem of Hain [2], see §5.

Theorem 2. *Suppose M is a (not necessarily compact) connected smooth complex algebraic variety, G is a reductive algebraic group defined over $\mathbb{R}$ and $\rho : \pi_1(M) \to G$ is a representation with finite image. Then the real-analytic germ $(\mathrm{Hom}(\pi_1(M), G), \rho)$ (and its complexification) is a quasi-homogeneous cone with generators of weights 1 and 2 and relations of weights 2, 3 and 4. Suppose further that there is a local cross-section through ρ to the $Ad(G)$-orbits in $\mathrm{Hom}(\pi_1(M), G)$. Then the quotient real-analytic germ $(\mathrm{Hom}(\pi_1(M), G)//G, [\rho])$ (and its complexification) is a quasi-homogeneous cone with generators of weights 1 and 2 and relations of weights 2, 3 and 4.*

Here we use the following

Definition. *Let X be a real or complex analytic space $x \in X$ and G a Lie group acting on X. We say that there is a* **local cross-section** *through x to the G-orbits if there is a G-invariant open neighborhood U of x and a closed analytic subspace $S \subset U$ such that the natural map $G \times S \to U$ is an isomorphism of analytic spaces.*

2. Shephard and Artin groups. Let Λ be a finite graph where two vertices are connected by at most one edge, there are no loops (i.e. no vertex is connected by an edge to itself) and each edge e is assigned an integer $\epsilon(e) \geq 2$. We will assume further that Λ has no isolated vertices and each vertex v of Λ is labelled by an integer $\delta(v)$. We call Λ a *labelled* graph, let $\mathcal{V}(\Lambda)$ denote the set of vertices of Λ. Given Λ we construct two finitely-presented groups: the *Shephard group* G^s_Λ and the *Artin group* G^a_Λ. The sets of generators for the both groups are $\{g_v, v \in \mathcal{V}(\Lambda)\}$. Relations in G^a_Λ are:

$$\underbrace{g_v g_w g_v g_w \cdots}_{\epsilon \text{ multiples}} = \underbrace{g_w g_v g_w g_v \cdots}_{\epsilon \text{ multiples}}, \quad \epsilon = \epsilon(e), \text{ over all edges } e = [v, w]$$

For example, if Λ has n vertices and no edges then G^a_Λ is the free group on n generators; if Λ is the complete graph on n vertices and $\epsilon(e) = 2$ for all e, then G^a_Λ is the free abelian group of rank n. To get the presentation for G^s_Λ we add the relations: $g_v^{\delta(v)} = \mathbf{1}$, $v \in \mathcal{V}(\Lambda)$. There is a canonical epimorphism $G^a_\Lambda \to G^s_\Lambda$.

3. Arrangements. An *abstract arrangement* A is a disjoint union of two finite sets $A = \mathcal{P} \sqcup \mathcal{L}$ with the set of "points" $\mathcal{P} = \{v_1, v_2, \dots\}$ and the set of "lines" $\mathcal{L} = (l_1, l_2, \dots)$, together with the incidence relation $\iota = \iota_A \subset \mathcal{P} \times \mathcal{L}$; $\iota(v, l)$ is interpreted to mean "the point v lies on the line l ". We may represent the arrangement A by a bipartite graph $\Gamma = \Gamma_A$. An arrangement is called *admissible* if it satisfies the axiom:

Every vertex in the graph Γ is incident to at least two edges.

An example of an abstract arrangement is the *standard triangle* (or *complete quadrilateral*) T (see Figure 1 where we draw *points* of A as solid points and *lines* as lines). An abstract *based arrangement* A is an arrangement together with an embedding $T \hookrightarrow A$ of the standard triangle.

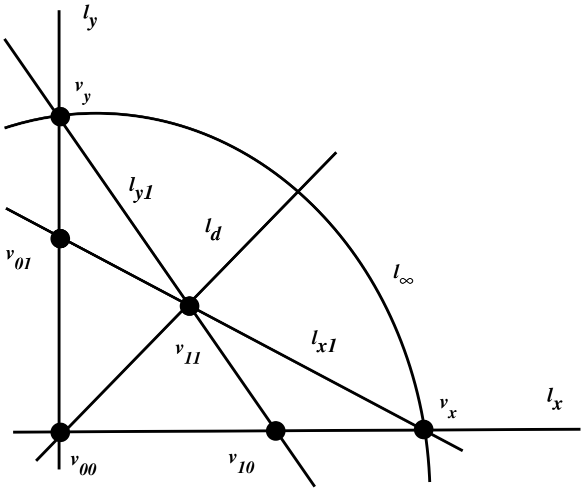

Figure 1: *The standard triangle T. (Triangle standard T.)*

We consider the projective plane $\mathbb{P}^2$ over a field $\mathbf{k}$ of zero characteristic and $(\mathbb{P}^2)^\vee$, the space of lines in $\mathbb{P}^2$. A *geometric realization* of the abstract arrangement $A = \mathcal{P} \sqcup \mathcal{L}$ is a map $\phi : \mathcal{P} \sqcup \mathcal{L} \to \mathbb{P}^2 \sqcup (\mathbb{P}^2)^\vee$ which sends *points* to points and *lines* to lines (we also use the term *projective arrangement* for ϕ). This map must satisfy the following condition:

$$\iota(v, l) \;\Rightarrow\; \widetilde{\phi(l)}(\widetilde{\phi(v)}) = 0 \tag{1}$$

Here we have lifted the line $\phi(l)$ to $\widetilde{\phi(l)} \in (\mathbf{k}^3)^\vee - \{0\}$ and the point $\phi(v)$ to $\widetilde{\phi(v)} \in \mathbf{k}^3 - \{0\}$.

For the standard triangle T define a realization ϕ_T of T in $\mathbb{P}^2$ so that the images of the *points* v_{00}, v_x, v_y, v_{11} have the homogeneous coordinates $(0:0:1), (1:0:0), (0:1:0), (1:$

$1:1)$ respectively. This realization extends uniquely to the rest of T. The *configuration space* of an abstract arrangement A is the space $R(A)$ of all geometric realizations of A. The space $R(A)$ is the set of $\mathbf{k}$-points of a projective variety defined over $\mathbb{Z}$ with equations determined by the condition (1). A *based realization* is a realization ϕ of a based abstract arrangement A such that the restriction of ϕ to the canonically embedded triangle T is the standard realization ϕ_T. The space $BR(A)$ of based realizations of an arrangement A is a projective variety defined over $\mathbb{Z}$. The variety $BR(A)$ is naturally isomorphic to a Mumford quotient $R(A)//PGL(3)$ (for a certain choice of projective embedding of $R(A)$). Suppose that A is a based arrangement which has a distinguished set of *points* $\nu = \{v_1, ..., v_n\}$ which lie on the *line* l_x. We call A_ν a *marked* arrangement. Let $BR_0(A_\nu) := \{\psi \in BR(A) : \psi(v_j) \notin L_\infty\ ,\ v_j \in \nu\}$ where $L_\infty = \phi_T(l_\infty)$. In [4] we prove:

Theorem 3. *For any affine scheme S of finite type over $Spec(\mathbb{Z})$, realized as a closed subscheme of affine space $\mathbb{A}^n$, there is a based arrangement marked by $\nu = \{v_1, ..., v_n\}$, such that the map $(\phi(v_1), ..., \phi(v_n))$ from $BR_0(A)$ to $\mathbb{A}^n$ induces an isomorphism τ of $BR_0(A)$ to S.*

Remark. *The arrangement A is not uniquely determined by the affine embedding of S, it also depends on equations defining S and their description via compositions of elementary algebraic operations.*

A version of this theorem appears in Mnev's paper [6], where he asserts only a *stable homeomorphism* between the sets of real points of $BR_0(A)$ and S and gives an outline of the proof. As a corollary of Theorems 1 and 3 and [1, Corollary 2.8.6] we get:

Corollary. *For every smooth connected compact manifold M there exists an abstract arrangement A and an Artin group G^a_A so that the manifold M is diffeomorphic to the set of real points in the moduli space $R(A)//PGL(3)$ and to a component (in the classical topology) in* $\mathrm{Hom}(G^a_A, SO(3,\mathbb{R}))/SO(3,\mathbb{R})$.

4. The associated representation varieties. We will define several classes of groups corresponding to abstract arrangements. Let $\Gamma = \Gamma_A$ be the bipartite graph of a based abstract arrangement. We introduce the new edges $[v_{10}, v_{00}]$, $[v_{01}, v_{00}]$, $[v_{11}, v_{00}]$ and identify the vertices: $v_{00} \cong l_\infty$, $v_{\infty 0} \cong l_y$, $v_{0\infty} \cong l_x$; let Λ be the resulting graph. Put the following labels on the edges of Λ:

Assign the label 4 to the edges $[v_{10}, v_{00}], [v_{01}, v_{00}]$ and all the edges which contain v_{11} as a vertex (with the exception of $[v_{11}, v_{00}]$). We put the label 6 on the edge $[v_{11}, v_{00}]$. Assign the label 2 to the rest of the edges. Put the label 3 to the vertex v_{11} and labels 2 to the rest of the vertices. Now we have labelled graphs and we use the procedure from the Section 2 to construct the *Artin group* $G^a_A := G^a_\Lambda$ and the *Shephard* group $G^s_A := G^s_\Lambda$.

Let q be the quadratic form $x_1^2 + x_2^2 + x_3^2$ and $PO(3)$ be the projectivized group of isometries of q. From now on we work over $\mathbb{Q}$ (rather than $\mathbb{Z}$). Let Z be the projectivized null quadric of q and $\mathbb{P}^2_0 = \mathbb{P}^2 - Z$. We let $(\mathbb{P}^2_0)^\vee$ be the image of $\mathbb{P}^2_0$ under the polarity defined by q. A projective arrangement ψ will be said to be *anisotropic* if $\psi(v) \in \mathbb{P}^2_0, \psi(l) \in (\mathbb{P}^2_0)^\vee$, for all $v \in \mathcal{P}, l \in \mathcal{L}$. The anisotropic condition defines Zariski open subsets of the previous arrangement varieties, to be denoted $R(A, \mathbb{P}^2_0), BR(A, \mathbb{P}^2_0)$ and $BR_0(A, \mathbb{P}^2_0)$ respectively.

Now a point P in $\mathbb{P}^2_0$ determines the Cartan involution σ_P in $PO(3, \mathbf{k})$ around this point or the rotation θ_P of order 3 having P as neutral fixed point (i.e. a point where the differential of rotation has the determinant 1). There are two such rotations of order 3, we choose one of them. Since ψ is *based*, $\psi(v_{11}) = (1:1:1)$ for all ψ, hence the choice of rotation is harmless (see [4, §12.1]). Similarly a line $L \in (\mathbb{P}^2_0)^\vee$ uniquely determines the reflection σ_L which keeps L pointwise fixed. Finally one can encode the incidence relation between points and lines in $\mathbb{P}^2$ using algebra: two involutions generate the subgroup $\mathbb{Z}/2 \times \mathbb{Z}/2$ in $PO(3, \mathbf{k})$ iff the neutral fixed point of one belongs to the fixed line of another, rotations σ, θ of orders 2 and 3 anticommute (i.e. $\sigma\theta\sigma\theta = 1$) iff the neutral fixed point of the rotation θ belongs to the fixed line of the involution σ, etc. We get the *algebraization* morphism:

$$alg: \ \text{based anisotropic arrangements} \quad \longrightarrow \quad \text{representations}$$

$$alg: \psi \mapsto \rho \ , \quad \rho(g_v) = \sigma_{\psi(v)} \ , \ v \in \mathcal{V}(\Lambda) - \{v_{11}\} \ \ , \ \ \rho(g_{v_{11}}) = \theta_{\psi(v_{11})} \ ,$$
$$\rho \in \mathrm{Hom}(G^s_A, PO(3)) \ , \quad \psi \in BR(A)$$

Theorem 4. *The mapping $alg: BR(A, \mathbb{P}^2_0) \to X(G^s_A\ , PO(3))$ is an isomorphism onto a Zariski open and closed subvariety to be denoted $X^+_f(G^s_A\ , PO(3))$.*

The mapping alg has the following important property: let S be an affine scheme of finite type over $\mathbb{Z}$ and $O \in S$ be an integer point. Choose an embedding (defined over $\mathbb{Z}$) $\tilde{S}$ of S into affine space such that O goes to the origin. We can define $\tilde{S}$ via equations that do not contain multiplicative and additive constants (so for instance the equation $2x + 1 = 1$ will be rewritten as $x + x = 0$). For $\tilde{S}$ choose a suitable arrangement A as in Theorem 3 and let $\psi_0 \in BR_0(A)$ correspond to the origin under the isomorphism $\tau: BR_0(A) \to \tilde{S}$.

Lemma. *The image of G^s_A under $alg(\psi_0)$ is finite.*

It remains to examine the morphism $\mu: X^+_f(G^s_A, PO(3)) \to X(G^a_A, PO(3))$ given by pull-back of homomorphisms to $PO(3)$.

Theorem 5. *Suppose that A is an admissible based arrangement. Then the morphism μ is an isomorphism onto a union of Zariski connected components.*[1]

Corollary 6. *The character variety $X(G^a_A, PO(3))$ inherits all the singularities of the character variety $X(G^s_A, PO(3))$ corresponding to points of $BR(A, \mathbb{P}^2_0)$.*

5. Hain's theorem.

Theorem 7. *Suppose M is a connected smooth complex algebraic variety, G is a reductive algebraic group defined over $\mathbb{R}$ and $\rho: \pi_1(M) \to G$ is a representation with finite image. Then the twisted de Rham algebra $A^\bullet(M, ad\rho)$ and its complexification $A^\bullet(M, ad\rho) \otimes \mathbb{C}$ have the structure of a mixed Hodge complex. The graded bracket is compatible with Hodge and weight filtrations. The first cohomology has the weights (at most) 1 and 2 and the second cohomology has the weights (at most) $2, 3$ and 4.*

We recall [5] that there is a complete local ring $R_{L^\bullet}$ associated to any differential graded Lie algebra $L^\bullet$. Suppose that there exists a local cross-section through ρ to the $Ad(G)$-orbits in $\mathrm{Hom}(\pi_1(M), G)$. Then by [3], Theorem 2.3, the complete local ring associated to the quotient germ $(\mathrm{Hom}(\pi_1(M), G)//G, [\rho])$ is isomorphic to $R_{L^\bullet}$ where $L^\bullet$ is the twisted de Rham algebra $A^\bullet(M, ad\rho)$. For $\rho = alg(\psi_0)$ the required local cross-section is shown to exist in [4, §12.1]. Finally to prove Theorem 2 we use:

Theorem 8. (Hain's Theorem) *Suppose $L^\bullet$ and $L^\bullet \otimes \mathbb{C}$ have the structure of a mixed Hodge complex compatible with the graded bracket and $\mathrm{H}^0(L^\bullet) = 0$. Then there is a morphism of graded (by weight) vector spaces*

$$\delta: \mathrm{H}^2(L^\bullet_{\mathbb{C}})^* \to Gr^W \mathbb{C}[[\mathrm{H}^1(L^\bullet_{\mathbb{C}})^*]]$$

with the image of δ contained in the square of the maximal ideal, so that $R_{L^\bullet_{\mathbb{C}}}$ is the quotient of $\mathbb{C}[[\mathrm{H}^1(L^\bullet_{\mathbb{C}})^*]]$ by the graded ideal generated by the image of δ.

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[1] The Zariski open subsets U, U' in the abstract and Theorem 1 are $\tau(BR_0(A, \mathbb{P}^2_0))$ and $\mu \circ alg(BR_0(A, \mathbb{P}^2_0))$ respectively.

Here Gr^W denotes the graded vector space associated to the filtered (by weight) vector space $\mathbb{C}[[\mathrm{H}^1(L^\bullet_{\mathbb{C}})^*]]$. We also give an alternative proof of Theorem 2 using results of Morgan [7] on minimal models of smooth complex algebraic varieties.

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